Endogenous Liquidity and Defaultable Bonds*

Zhiguo He† University of Chicago and NBER

Konstantin Milbradt‡ MIT Sloan

January 31, 2013

Abstract

This paper studies the interaction between default and liquidity for corporate bonds that are traded in an over-the-counter secondary market with search frictions. Bargaining with dealers determines a bond's endogenous liquidity, which depends on both the firm fundamental and the time-to-maturity of the bond. Corporate default decisions interact with the endogenous secondary market liquidity via the rollover channel. A default-liquidity loop arises: Earlier endogenous default worsens a bond's secondary market liquidity, which amplifies equity holders' rollover losses, which in turn leads to earlier endogenous default. Besides characterizing in closed form the full inter-dependence between liquidity premium and default premium for credit spreads, we also also calibrate the model to jointly match empirically observed credit spreads and liquidity measures of bonds across different rating classes.

Keywords: Positive Feedback, Fundamental and Liquidity, Over-The-Counter Market, Secondary Bond Market, Structural Models for Credit Risk, Transaction Cost for Corporate Bonds, Bid-Ask Spread

*For helpful comments, we thank Nittai Bergman (MIT), Bruce Carlin (UCLA), Hui Chen (MIT), Richard Green (CMU), Nicolae Garleanu (UC Berkeley), Barney Hartman-Glaser (Duke), Burton Hollifield (CMU), Gustavo Manso (UC Berkeley), Holger Mueller (NYU), and seminar participants of the MIT Sloan lunchtime workshop, NYU lunchtime workshop, Columbia GSB lunchtime workshop, NBER Microstructure meeting, ASU winter conference, Duke-UNC asset pricing conference, Texas Finance Festival, UNC, Boston University, University of Colorado at Boulder, INSEAD, Imperial College London, UCLA Anderson, WFA 2012, SED 2012, NBER SI Asset Pricing meeting, and Gerzensee ESSFM 2012. We are especially grateful to Rui Cui for excellent research assistance. Zhiguo He acknowledges financial support from the Center for Research in Security Prices at the University of Chicago Booth School of Business.

†Booth School of Business, University of Chicago, and NBER. 5807 South Woodlawn Ave., Chicago, IL 60637; e-mail: zhiguo.he@chicagobooth.edu.

‡MIT Sloan School of Management, 100 Main Street E62-633, Cambridge, MA 02142; e-mail: milbradt@mit.edu.

1 Introduction

The recent 2007-2008 financial crisis and the ongoing sovereign crisis have vividly demonstrated the intricate interaction between asset fundamental and asset liquidity in financial markets. Liquidity tends to dry up for assets with deteriorating fundamentals when solvency becomes a concern, reflected by soaring liquidity premia and/or prohibitive transaction costs in trading. In the mean time, asset fundamentals worsen further due to endogenous reactions (say, default) of market participants in response to worsening liquidity in financial markets.

The default-liquidity spiral is at the center of the academic policy research on financial crises, and this paper aims to deliver such a feedback loop in the context of corporate bond markets.[1] It has been well documented that secondary corporate bond markets – which are mainly over-the-counter (OTC) markets – are much less liquid than equity markets.[2] On the one hand, Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011) document a strong empirical pattern that the liquidity for corporate bonds (measured as the transaction cost) deteriorates dramatically for bonds with lower fundamental, i.e., bonds that are issued by firms closer to default (reflected by higher credit derivative swaps, CDS). On the other hand, the recent financial crisis of 2007-2008 illustrates that the deterioration of secondary market liquidity, through adversely affecting the refinancing operations of firms, can exacerbate the incentives of equity holders to default. Taken together, these two observations imply a positive feedback loop between the secondary market liquidity and asset fundamentals for corporate bonds. Indeed, recent research, e.g., Dick-Nielsen, Feldhutter, and Lando (2012), and Friewald, Jankowitsch, and Subrahmanyam (2012), shows that liquidity in corporate bond market dries up substantially during the 2007/2008 crisis, and much more so for bonds with speculative grade.

[1]Corporate bond markets, for both financial and non-financial firms, make up a large part of the U.S. financial system. According to flow of funds, the values of corporate bonds reaches about 4.7 trillion in the first quarter of 2010, which consists of about one third of total liabilities of U.S. corporate businesses.

[2]For instance, Edwards, Harris, and Piwowar (2007) study the U.S. OTC secondary trades in corporate bonds and estimate the transaction cost to range from 30 to 100 bps, and Bao, Pan, and Wang (2011) find an even larger number. The fact that equity markets–while being presumably subject to more asymmetric information problems–are more liquid imply the importance of search friction in corporate bond markets. Other empirical papers that investigate secondary bond market liquidity are Hong and Warga (2000), Schultz (2001), Green, Hollifield, and Schurhoff (2007a,b); Harris and Piwowar (2006).

To deliver such a default-liquidity spiral effect, we adopt two standard ingredients from different existing literatures. First, we model the endogenous liquidity in the secondary corporate bond market as a search-based over-the-counter (OTC) market á la Duffie, Garlenau, and Pedersen (2005). Bond investors who are hit by liquidity shocks face holding costs for holding the asset and thus want to divest of it, and with a certain matching technology they meet and trade with an intermediary dealer at an endogenous bid-ask spread. A novel feature is that the endogenous liquidity for the secondary bond market depends on both the firm's distance-to-default and the bond's time-to-maturity.

The second important ingredient for the feedback between fundamental and liquidity is the endogenous default decision by equity holders. This mechanism is borrowed from the standard Leland-type corporate finance structural models, i.e., Leland and Toft (1996) (hereafter LT96). More specifically, a firm rolls over (refinances) maturing bonds by issuing new bonds of the same face value. When firm fundamentals deteriorate, equity holders will face heavier rollover losses due to falling prices of newly issued bonds. Equity holders default optimally when absorbing further losses is unprofitable, at which point bond investors with defaulted claims step in to recover part of the firm value subject to dead-weight bankruptcy cost.

The secondary market liquidity of *defaulted* bonds, i.e., bonds of firms that have defaulted, is important in deriving the endogenous bond liquidity before the firm defaults. We assume that bankruptcy leads to a delay in the payout of any cash due to lengthy court proceedings, and focus on the situation where the secondary market for defaulted bonds is more illiquid than that for non-defaulted bonds. In our model, we assume that the secondary market of non-defaulted bonds is a *seller's market* (i.e., more buyers than sellers); but once the firm defaults, as there are few potential buyers of defaulted securities, the secondary market for defaulted bonds turns to a *buyer's market* (more sellers than buyers), hurting sellers who are trying to sell their bond holdings. We solve for the post-default bond valuations in closed-form, and the above two forces lead to a greater valuation wedge at default between investors who have been hit by liquidity shocks and investors who have not.

The post-default bond valuations serve as the boundary conditions needed to solve the system of partial differential equations (PDEs) that describes the bond valuations before the firm defaults.[3] We solve for debt and equity valuation, the endogenous default boundary, and the endogenous liquidity in closed form in Section 3. Consistent with empirical findings in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011), we show in Section 3.4 that the endogenous bid-ask spread is decreasing with the firm's distance-to-default, holding the time-to-maturity constant.

The derived endogenous liquidity allows us to study the positive feedback loop between liquidity and default. Imagine an exogenous negative cash flow shock that pushes the firm closer to default, lowering the bond's fundamental value. More importantly, because bonds of defaulted firms suffer greater illiquidity, the outside option of the bond sellers when bargaining with the dealer declines. This worsens the secondary market liquidity and lowers the bond prices even further. The wider refinancing gap between the newly issued bond prices and promised principals gives rise to heavier rollover losses, which causes equity holders to default earlier and thus pushes the firm even closer to default. As a result, lower distance-to-default reduces the fundamental value of the corporate bonds even further, and so forth. The outcome of these spirals is a unique fixed point bankruptcy threshold at which equity holders default.

We investigate the quantitative performance of our model based on corporate bonds with different rating classes ranging from AAA to B. One great advantage of our corporate bond pricing model is that the secondary bond market liquidity endogenously depends on the firm's distance-to-default, which is one of key determinants for bonds across various ratings. Relative to the previous literature with exogenous secondary market liquidity (say He and Xiong (2012b)), our model ties the secondary market liquidity to firm's distance-to-default, and delivers desirable parsimony in that we can generate the empirical cross-sectional pattern of illiquidity across credit ratings by adjusting the firm's distance-to-default only. We choose holding cost parameters to target the observed bid-ask spread for bonds with investment BBB grade, and then calculate the model implied bid-ask

[3] This arises because bond valuations depend on firm fundamental, the bond's time-to-maturity, and the liquidity state of bond holders.

spreads for bonds in other ratings by matching their leverages. Our model fits the cross-sectional pattern observed in the data well, not only on bid-ask spreads but also on credit spreads.

Theoretically, our paper characterizes a full inter-dependence between liquidity and default components in the credit spread for corporate bonds. This contrasts with the widely-used reduced-form approach in the empirical research, where it is common to decompose firms' credit spreads into independent liquidity-premium and default-premium components (e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010)). Our fully solved structural model calls for more structural approaches in future studies on the impact of liquidity factors upon the credit spread of corporate bonds; indeed, our ongoing project suggests that our positive spiral may amplify small liquidity frictions into quantitatively significant liquidity and default premia.

Our paper belongs to the literature on the role of secondary market trading frictions in structural models of corporate finance (Black and Cox (1976), Leland (1994) and LT96). Ericsson and Renault (2006) analyze the interaction between secondary liquidity and the bankruptcy-renegotiation in a LT96 framework, and Duffie and Lando (2001) study credit risk when bond investors only have incomplete information. He and Xiong (2012b) take the simplified secondary market friction introduced in the classic article of Amihud and Mendelson (1986), i.e., bond investors hit by liquidity shocks are forced sell their holdings immediately at an exogenous and constant proportional transaction cost. Because in He and Xiong (2012b) the bond market liquidity is modeled in an exogenous way, that paper can only speak to the one-way economic channel from exogenous liquidity to default. In contrast, our paper endogenizes the secondary market liquidity by micro-founding the bond trading in a search-based secondary market, and derives the equilibrium liquidity *jointly* with equilibrium asset prices.[4]

[4]Another possibility to micro-found secondary market liquidity is to assume some adverse selection with regard to the bankruptcy recovery value, a path we do not pursue in this model due to the difficulties inherent in tracking persistent private information. Well-known endogenous market illiquidity models based on private information are Kyle (1985), Glosten and Milgrom (1985), and Back and Baruch (2004). Besides the advantage of being able to be integrated seamlessly into the dynamic firm setting in LT96, the search based framework is suitable for the secondary market for corporate bonds, especially considering the fact that equity markets have much higher liquidity while being subject to more severe asymmetric information problems. For adverse selection in search markets, see Lauermann and Wolinsky (2011) and Guerrieri, Shimer, and Wright (2010) (in directed search, rather than random as we assumed here).

Our paper also makes a contribution to the search based asset-pricing literature, as represented by Duffie, Garlenau, and Pedersen (2005, 2007); Weill (2007); Lagos and Rocheteau (2007, 2009); Biais and Weill (2009); Feldhutter (2011). To our knowledge, this literature with concentration on OTC markets has thus far focused on the determinants of contact intensities and behavior of intermediaries, while eschewing time-varying asset fundamentals. Undoubtedly, asset-specific dynamics are important for the corporate bonds market, and we fill this gap by incorporating the firm's distance-to-default and the bond's time-to-maturity in deriving the asset (bond) valuations.[5] Moreover, our paper demonstrates that, via the rollover channel, the endogenous search-based secondary market liquidity can have a significant impact on the firms' behavior on the real side.

Positive feedback is an active research topic in different areas. For instance, strategic complementarity naturally gives rise to positive feedback effect in the global games literature (Morris and Shin 2009), and a similar effect emerges in He and Xiong (2012a) who study dynamic coordinations among creditors whose debt contracts mature at different times. Our paper is more related to the literature that emphasizes the interaction between firms and financial markets. For example, Goldstein, Ozdenoren, and Yuan (2011) show that market prices can feedback to firm's investment decisions through the information channel; Brunnermeier and Pedersen (2009) illustrate the positive feedback loop between funding liquidity and market liquidity; Cheng and Milbradt (2012) show how managerial risk-shifting feeds back on bondholders decision to run, which in turn feeds back on managerial incentives; and Manso (2011) points out that credit ratings affect a firm's default decision, which feeds back into the rating decision.

Our paper is also related to the literature of debt maturity structure (Diamond, 1993, Leland, 1998, etc). For the use of short-term debt with a higher rollover frequency, there exists a trade-off between better liquidity provision and earlier inefficient default. Regarding the liquidity provision

[5] The existing literature often assumes infinite maturity and constant asset payoffs; for instance, focusing on a very different market, Vayanos and Weill (2008) use a search framework to explain the difference between off-the-run and on-the-run treasury yields. As far as we know, the only paper with deterministic time dynamics in a search framework is the contemporaneous Afonso and Lagos (2011), which introduces deterministic time dynamics via an end-of-day trading close in the federal funds market. More importantly, endogenous default with stochastic fundamental is one key building block for our paper. Because corporate bond payoffs are highly nonlinear in firm fundamentals, our closed-form solution with stochastic fundamentals is nontrivial.

of short-term debt, bond investors hit by liquidity shocks can either sell to dealers or sit out shocks by waiting to receive the face value when the bond matures. Shorter maturity improves upon the waiting option, resulting in a lower rent extracted by dealers and thus a greater secondary market liquidity. On the other hand, equity holders are absorbing rollover gains/losses ex post. As shown in LT96 and emphasized in He and Xiong (2012b), shorter-term debt with a higher rollover frequency leads to heavier rollover losses in bad times, which pushes equity holders to default earlier and thus to incur greater dead-weight bankruptcy costs. This tradeoff allows us to endogenize the firm's initial choice of debt maturity, and unlike traditional capital structure models an optimal finite maturity structure arises.

The paper is organized as follows. Section 2 lays out the model, and Section 3 solves the model in closed-form, and illustrates the positive feedback loop between fundamental and liquidity. We calibrate our model to match the cross-sectional pattern of bid-ask spread and credit spreads in Section 4. Section 5 provides extensions and discussions and Section 6 concludes. Proofs not given in the main text can be found in the Appendix.

2 The Model

We proceed by first describing the model from the firm's perspective. We then describe the individual bondholder's problem, and the secondary market where investors can trade their bond holdings.

2.1 Firm, Debt Rollover, and Endogenous Default

2.1.1 Firm Cash Flows and Debt Maturity Structure

We consider a continuous-time model where a firm has assets-in-place that generate (after-tax) cash flows at a rate of $\delta_t > 0$, where $\{\delta_t : 0 \le t < \infty\}$ follows a geometric Brownian motion under the risk-neutral probability measure:

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dZ_t, \tag{1}$$

where μ is the constant growth rate of cash flow rate, σ is the constant asset volatility, and $\{Z_t : 0 \leq t < \infty\}$ is a standard Brownian motion, representing random shocks to the firm fundamental. We assume the risk-free rate r to be constant in this economy.

We follow LT96 in assuming that the firm maintains a stationary debt structure. At each moment in time, the firm has a continuum of bonds outstanding with an aggregate principal of p and an aggregate coupon payment of c, where p and c are constants that we take as exogenously given. We normalize the measure of bonds to 1, so that each bond has a principal face value of p and a coupon flow payment of c. All bonds have an initial maturity T but differ in their current time-to-maturity $\tau \in [0, T]$. Expirations of the bonds are uniformly spread out across time;[6] that is, during a time interval $(t, t + dt)$, a fraction $\frac{1}{T}dt$ of the bonds matures and needs to be rolled over. Thus, $1/T$ is the firm's rollover frequency on its debt, and $\frac{T}{2}$ is the average maturity of the firm's outstanding bonds. We assume that the firm commits to a stationary debt structure denoted (c, p, T) in the following sense: when a bond matures, the firm will replace it by issuing a new bond with identical (initial) maturity T, principal value p, and coupon rate c, in the primary market.

This simple stationary debt maturity structure gives us a convenient dynamic setting to analyze the interaction between liquidity and default, and we believe the general economic mechanism identified in this paper is robust to this assumption. In the main analysis we take the firm's debt maturity T as given.[7]

[6]This staggered debt maturity structure is consistent with recent empirical findings (Choi, Hackbarth, and Zechner, 2012).

[7]One could also endogenize the firm initial leverage (c, p) based on the trade-off between tax benefit and bankruptcy cost by following LT96. We leave this exercise for future research.

2.1.2 Primary Bond Market and Debt Rollover

The firm replaces maturing bonds with newly issued ones of identical face-value in the so-called primary market, where the firm hires a dealer who can place the new debt to bond investors. Here we assume a constant proportional issuance cost $\kappa \in [0,1]$; this cost is unimportant for our main result. Per unit of newly issued bond, the firm receives the net proceeds of $(1-\kappa) D_H(\delta, T)$, where $D_H(\delta, T)$ is the primary market bond valuation given firm cash flow δ and time-to-maturity of T. The subscript of "H" will become clear when we describe the secondary market shortly. Issuance is via a one-price auction where we assume that the H-type agent is the marginal agent. Even though a higher valuation agent exits, called S-type, their mass is assumed too small to cover the whole bond issue.

Equity holders are the residual claimants of any rollover gains/losses. Following LT96, we assume that any gain will be immediately paid out to equity holders and any loss will be funded by issuing more equity at the market price. Thus, over a short time interval $(t, t+dt)$, the net cash flow to equity holders (omitting dt) is given by

$$NC_t = \underbrace{\delta_t}_{\text{CF}} - \underbrace{(1-\pi)\, c}_{\text{Coupon}} + \underbrace{\frac{1}{T}\left[(1-\kappa) D_H(\delta_t, T) - p\right]}_{\text{Rollover}}. \tag{2}$$

The first term is the firm's cash flow arising from its assets in place. The second term is the after-tax coupon payment to bond investors, where π denotes the marginal tax benefit rate of debt.[8] The third term captures the firm's rollover gains/losses by issuing new bonds to replace maturing bonds. This term can be understood as *repricing* the bonds at a rate of $1/T$. In this transaction, there is a $\frac{1}{T}dt$ fraction of bonds maturing, which requires a principal payment of $\frac{1}{T}pdt$; while the primary market value of the newly issued bonds is $\frac{1}{T}D_H(\delta_t, T)dt$, and the firm receives a fraction $(1-\kappa)$ of these proceeds. When the newly issued bond price $D_H(\delta_t, T)$ drops so that $(1-\kappa) D_H(\delta_t, T) < p$, equity holders have to absorb the negative cash-flow stemming from rollover

[8] For each dollar received by bond investors, the government is subsidizing π dollars so that equity holders only have to pay $1-\pi$ dollars. The tax advantage of debt π affects the equity holders' endogenous default decision.

$\frac{1}{T}[(1-\kappa)\, D_H(\delta_t, T) - p]dt$. Thus, the rollover frequency $1/T$ (or the inverse of debt maturity) affects the extent of rollover losses/gains.

2.1.3 Bankruptcy

When the firm issues additional equity to fund these rollover losses, the equity issuance dilutes the value of existing shares.[9] Equity holders are willing to buy more shares and bail out the maturing debt holders as long as the equity value is still positive (i.e. the option value of keeping the firm alive justifies absorbing the rollover losses), and drops to zero when the firm defaults at $\delta = \delta_b$. Then, creditors can only recover a fraction α of the firm's unlevered value from liquidation,[10] and for simplicity we assume equal seniority of all creditors. Equity holders optimally pick δ_b, which will be an important ingredient for the feedback loop between default and secondary market liquidity.[11]

So far the model followed the standard assumptions in the structural debt literature. Now we introduce a new element. As will become clear shortly, we model "liquidity" based on the idea that agents may be subject to holdings costs while in possessions of the bond. Hence, our bankruptcy treatment has to be careful in this regard. If bankruptcy leads investors to receive the bankruptcy proceeds immediately in exchange for the bond, creditors may view default as a beneficial outcome.[12] This "liquidity by default" runs counter to the fact that in practice bankruptcy leads to a more illiquid secondary market, the freezing of assets within the company, and a delay in the payout of any cash depending on court proceedings.[13]

[9] A simple example works as follows. Suppose a firm has 1 billion shares of equity outstanding, and each share is initially valued at \$10. The firm has \$10 billion of debt maturing now, but the firm's new bonds with the same face value can only be sold for \$9 billion. To cover the shortfall, the firm needs to issue more equity. As the proceeds from the share offering accrue to the maturing debt holders, the new shares dilute the existing shares and thus reduce the market value of each share. If the firm only needs to roll over its debt once, then the firm needs to issue 1/9 billion shares and each share is valued at \$9. The \$1 price drop reflects the rollover loss borne by each share.

[10] The bankruptcy cost is standard in the trade-off literature, and can be interpreted in different ways, such as loss of customers or or legal fees. However, as we will introduce inefficient delay in court rulings shortly, our analysis goes through even if there is no bankruptcy cost, i.e., $\alpha = 1$.

[11] To focus on the liquidity effect originating from the debt market, we ignore any additional frictions in the equity market such as transaction costs and asymmetric information. It is important to note that while we allow the firm to freely issue more equity, the equity value can be severely affected by the firm's debt rollover losses. This feedback effect allows the model to capture difficulties faced by many firms in raising equity during a financial-market meltdown even in the absence of any friction in the equity market.

[12] This would be the case for example for a CDS contract written on the firm which features immediate payouts at the time of a bankruptcy/credit event.

[13] The Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty,

Motivated by these facts, we make the following assumption for defaulted bonds. Suppose that ultimate bankruptcy recovery is based on the unlevered firm value at the time of default, $\frac{\delta_b}{r-\mu}$.[14] To capture the uncertain timing of the court decision, we introduce a court delay so that the payout of cash $\alpha\frac{\delta_b}{r-\mu}$ occurs at a Poisson arrival time with intensity θ. We focus on situations where $\alpha\frac{\delta_b}{r-\mu} < p$ so that the recovery rate to bond holders is below 100%.

2.2 Secondary Bond Market and Search-Based Liquidity

In this section we describe the structure of search-based secondary market for corporate bonds. All bond transactions are intermediated by dealers who form a competitive inter-dealer market. Throughout, following Duffie, Garlenau, and Pedersen (2005) we simply assume that each bond investor can at most hold one unit of the bond, while dealers cannot hold any inventory.[15]

2.2.1 Liquidity Shocks

As in Duffie, Garlenau, and Pedersen (2005), individual bond investors are subject to idiosyncratic liquidity shocks, and once hit by shocks they need to search for dealers to trade with. More specifically, before bankruptcy an individual bond holder is hit by an idiosyncratic liquidity shock with intensity ξ which is i.i.d. from all other bond holders. We model this sudden need for liquidity as a holding cost of $\chi_0 p + \chi_1 c$ where c is the coupon payment and p is the principal. It is a priori unclear whether the holding cost after a liquidity shock should be proportional to coupon or principal, and this modeling allows for more flexibility in calibration. For simplicity, this liquidity status lasts until either the agent manages to sell the bond, or until the bond matures. After either event, the investor exits the market forever.[16] Liquidity shocks can also occur after the firm defaults; here, we simply assume that the investor hit by a liquidity shock incurs a holding cost that is proportional to the final bankruptcy payout to be specified in detail below. We call those

payouts to the debt holders only started trickling out after about three and a half years.

[14]This is wlog and the recovery value could be made stochastic at the cost of more complicated solutions.

[15]Allowing for bilateral, i.e., non-intermediated, sales would lead to a system of Integro-Differential PDEs, something we are not be able to solve. However, such bilateral sales would not expand the economic mechanism of the model.

[16]This assumption is for easier exposition, and can easily be relaxed as shown in the Appendix.

non-liquidity-shocked investors H type investors (i.e., high valuation agent), and those liquidity-shocked investors L type investors (i.e., low valuation agent). The individual liquidity shock is uninsurable and thus results in an incomplete market and type dependent valuations, as explained below.

2.2.2 Bond Investors and Firm Default

It is important to have different secondary market structures for bonds that have not yet defaulted and for bonds that are in default. Indeed, in our model one of the channels that the firm's default affects secondary market liquidity is through the assumption that a significant population of (potential) bond investors are not allowed to buy defaulted bonds.

We assume that there are two classes of H type investors who are ready to buy bonds from dealers, and they differ in how default affects their preference/ability of investing in corporate bonds. More specifically, one class of investors is sensitive to the default event. They have a liquidity shock intensity of ξ before default; after default, their liquidity shock intensity goes up to $\xi_b > \xi$. Additionally they cannot purchase defaulted bonds. Without risk of confusion, we keep referring to these investors as H types. In contrast, there is another class of investors, denoted by S (for Specialists), who have a constant liquidity shock intensity ξ independent of whether the bond has defaulted or not, and are able to buy bonds pre- and post-default.

The classes of H and S represent different institutional buyers of corporate bonds in practice. The class of H investors represents normal corporate bond funds (say, money market funds, high yield bond funds, etc) who can only invest in bonds that have not defaulted yet, while the class of S investors represent hedge funds that are specializing in buying defaulted bonds and waiting for recovery. Our modeling of H investors and S investors captures this important difference in the most stark way, and we make an assumption below about the relative mass between H investors of S investors to reflect the scarcity of hedge funds who specialize in distressed securities. Interestingly, under that assumption, we will show that the valuation of S type investors does not affect the pre-default equilibrium outcome.

For simplicity, we assume that after purchasing the bond, if a liquidity shock hits either an H or S investors, both transit to the same L type.

2.2.3 Dealers and Competitive Dealer Market

In practice, secondary corporate bond markets are less liquid than equity or primary debt markets. Thus, we assume that the secondary debt markets are subject to the following trading friction. An L bond investor who wants to sell his debt-holdings has to wait an exponential time with intensity λ to meet a dealer. Similarly, an H or S investor who wants to buy has to wait an exponential time with intensity λ to meet a dealer. When they meet, bargaining occurs over the economic surplus generated. We follow Duffie, Garlenau, and Pedersen (2007) and assume Nash-bargaining weights β of the investor and $(1-\beta)$ of the dealer to model this bargaining. For simplicity, we assume the same bargaining power allocation for all dealer-investor pairs.

The illiquidity of secondary bond markets give rise to wedges in bond valuations for different investor types. Define $D_S(\delta,\tau)$, $D_H(\delta,\tau)$ and $D_L(\delta,\tau)$ to be the valuations of the S type, the H type, and the L type, respectively, for holding one unit of the bond. Suppose that a contact between a type L investor and a dealer occurs. As in Duffie, Garlenau, and Pedersen (2005), a dealer faces a frictionless competitive inter-dealer market with a continuum of dealers: a dealer in contact with an L investor can instantaneously sell a bond at a price M to another dealer who is in contact with an H or S investor. If he does so, the bond travels from an L investor to an H or S investor via the help of the two dealers who are connected in the inter-dealer market. Let us denote by B the bid price at which the L type is selling his bond, and by A (A^S) the ask price at which the H type (S type) is purchasing this bond.

2.3 Equilibrium Bond Transaction Prices

Throughout, we will impose the following assumptions about the relative aggregate buy/sell flows coming to the inter-dealer market. It is beyond the scope of this paper to endogenize the relative investors flows given the partial equilibrium nature of this paper.

Assumption 1 *Before default, the flow of L type sellers in contact with dealers is greater than the flow of S type buyers in contact with dealers, but smaller than the flow of H type buyers in contact with dealers, a situation we denote by the term seller's market. After default, H type investors withdraw from the buy side, and the flow of L type sellers in contact with dealers is smaller than the flow of S type buyers in contact with dealers, a situation we term a buyer's market.*

As explained before, Assumption 1 captures the key idea that firm default greatly reduces the potential investor base for corporate bonds. As we will derive below, Assumption 1 implies that the marginal buyer before default is an H type investor, whereas after default the marginal buyer is an S type investor.

2.3.1 Post-Default Bid-Ask Prices and Debt Valuations

We first analyze the secondary market for defaulted bonds, which is a direct application of the equilibrium with a competitive inter-dealer market in Duffie, Garlenau, and Pedersen (2005). We will use "b" to indicate the state of bankruptcy. Recall that after default, only S investors are able to purchase bonds, H investors without bond holdings withdraw from the market, and H investors with bond holdings are hit by liquidity shocks at an intensity $\xi_b > \xi$. Thus, relative to the market before default where there is always sufficient H type buyers to meet the supply from L type sellers, buy orders drop abruptly and selling pressure increases. In other words, the post default market is a *buyer's market.*

Denote the post-default debt valuation for H (L) investors by $D_H^{b,i}$ ($D_L^{b,i}$), where the index $i \in \{0, 1\}$ indicates the investors' holding. Clearly, the continuation values are $D_H^{b,0} = D_L^{b,0} = 0$ because H type investors cannot buy defaulted bonds, and L type investors exit after selling their bonds. So from now on we simply use D_H^b (D_L^b) to denote $D_H^{b,1}$ ($D_L^{b,1}$). For S investors, we denote their values by $D_S^{b,i}$, where the index $i \in \{0, 1\}$ indicates the holding of the S investor. $D_S^{b,0} \geq 0$ because S investors provide liquidity to the market and thus earn weakly positive rents in equilibrium.

The following lemma gives the equilibrium bid and ask prices as well as inter-dealer market price in the post-default equilibrium. Under Assumption 1, in the post default secondary market, some dealer-L type sellers are rationed without selling their holdings. When an L investor sells to a dealer who then sells the bond for a price M on the competitive inter-dealer market, the surplus is given by

$$\Pi_L^b = \left[D_L^b - D_L^{b,0}\right] - M^b = D_L^b - M^b.$$

Because the inter-dealer market is competitive in the Bertrand sense, the equilibrium inter-dealer market price $M^b = D_L^b$ so that $\Pi_L^b = 0$. Otherwise, if $M^b < D_L^b$ so that $\Pi_L^b > 0$, a dealer-L type pair could lower their selling price, be assured a sale and still receive positive surplus. But by the nature of Bertrand competition, no surplus can remain on the abundant side of the market. Zero surplus also implies that the equilibrium bid price is $B^b = D_L^b$.

As the buy side is made up of dealer-S type pairs, define the surplus from trade for an S type as $\Pi^b \equiv \Pi_S^b \equiv D_S^{b,1} - D_S^{b,0} - D_L^b > 0$. Following Nash-bargaining, the ask price at which S types (with a bargaining power of β_S) buy from the dealer, is given by $A^b = D_L^b + (1 - \beta_S)\,\Pi^b$. Thus, a *buyer's market* is characterized by positive surplus from trade for buyers, and zero surplus from trade for sellers.

The following lemma summarizes our findings.

Lemma 1 *Under Assumption 1, in the post-default market, we have*

$$\begin{aligned} A^b &= D_L^b + (1 - \beta_S)\,\Pi^b, \\ B^b &= M = D_L^b. \end{aligned}$$

Now we solve for the equilibrium values in the secondary market for defaulted bonds. Recall that the bankruptcy payout occurs with intensity θ. Further assume that post-default holding costs are proportional the the ultimate recovery payout, $\chi_b \frac{\delta_b}{r-\mu}$. We have the following linear system:

$$
\begin{aligned}
rD_H^b &= 0 + \xi_b \left(D_L^b - D_H^b \right) + \theta \left(\alpha \frac{\delta_b}{r-\mu} - D_H^b \right), && (3) \\
rD_L^b &= -\chi_b \frac{\delta_b}{r-\mu} + 0 + \theta \left(\alpha \frac{\delta_b}{r-\mu} - D_L^b \right) && (4) \\
rD_S^{b1} &= 0 + \xi \left(D_L^b - D_S^{b1} \right) + \theta \left(\alpha \frac{\delta_b}{r-\mu} - D_S^{b1} \right) \\
rD_S^{b0} &= 0 + \lambda_S \left(D_S^{b1} - A - D_S^{b0} \right) + \theta \left(0 - D_S^{b0} \right)
\end{aligned}
$$

On the right hand side of (4), the first term is the holding cost $\chi_b \frac{\delta_b}{r-\mu}$. The second term captures the value increment in contacting the dealer successfully; but it is zero because L investors sell their bond always at their reservation price D_L^b.

Proposition 2 *Under Assumption 1, post default the debt valuations for H and L investors are*

$$
\begin{bmatrix} D_H^b(\delta_b) \\ D_L^b(\delta_b) \end{bmatrix} = \begin{bmatrix} \alpha_H \\ \alpha_L \end{bmatrix} \frac{\delta_b}{r-\mu} \tag{5}
$$

where

$$
\alpha_H = \frac{\theta\alpha}{r+\theta} - \frac{\xi_b \chi_b}{(r+\theta)(r+\theta+\xi_b)}, \text{ and } \alpha_L = \frac{\theta\alpha}{r+\theta} - \frac{\chi_b}{r+\theta}.
$$

The values of $D_H^b(\delta_b)$ and $D_L^b(\delta_b)$ will serve as our boundary conditions for solving bond valuation functions before the firm defaults. Also, there is a wedge $\alpha_H - \alpha_L = \frac{\chi}{r+\theta+\xi_b} > 0$ in bankruptcy recoveries for H and L investors, a result that we will use later.

2.3.2 Pre-default Bid-Ask Prices

We now analyze the secondary market before the firm has defaulted. Denote the value functions of an H (L) investor without bond holding by D_H^0 (D_L^0). As L investors exit the market after selling, $D_L^0 = 0$ by assumption.

Note that an S investor's surplus is always weakly higher than an H investor's one. Under Assumption 1, however, in equilibrium the marginal buyer is an H type, as we are in a *seller's* *market* with an oversupply of H type buyers and an undersupply of S type buyers. The surplus generated from an H investors buying from a dealer who acquires the asset for a price M on the competitive inter-dealer market is given by

$$\Pi_H = \left[D_H - D_H^0\right] - M$$

Since some dealer-H type pairs have to be rationed without purchase due to too many willing buyers, Bertrand competition in the inter-dealer market drives the surplus Π_H to zero. Suppose not and $\Pi_H > 0$. Thus, $M < \left(D_H - D_H^0\right)$, and a dealer-$H$ type pair could offer a slightly higher price $M' > 0$ with $0 < \Pi_H' < \Pi_H$ and be assured a trade, thereby ensuring a positive profit. But this cannot be with an oversupply of buyers. Note further that $\Pi_H = 0$ also implies that $D_H^0 = 0$, because there is no other benefit accruing to an H investor without any current holding. Consequently, $D_H^0 = 0$ and we have $M = D_H$. Nash bargaining then results in the ask price $A = (1-\beta)\,\Pi_H + M = D_H$.[17]

On the sell side, the dealer-L type bargaining determines the bid price B. As a dealer can instantaneously sell at $M = D_H$ through the inter-dealer market, the surplus from trade is

$$\Pi \equiv \Pi_L \equiv M + \left[D_L^0 - D_L\right] = D_H - D_L > 0.$$

The transaction price at which L types sell to the dealer, B, thus implements the following splits of the surplus according to the bargaining weights is

$$B(\delta,\tau) = \underbrace{\beta \cdot \Pi(\delta,\tau)}_{\text{Appropriated surplus}} + \underbrace{D_L(\delta,\tau)}_{L\text{type's outside option}}. \tag{6}$$

Thus, a *seller's market* is characterized by a positive surplus from trade for the seller, and zero surplus from trade for the buyer. We summarize our findings in the Proposition below.

[17] Although the H type investors have some positive bargaining power, the presence of a competitive inter-dealer market erodes any surplus this bargaining power could extract.

Proposition 3 *Fix valuations $D_S > D_H$ and D_L. Under Assumption 1, the (lowest) ask price A and inter-dealer market price M are equal to D_H, and the bid price is given by $B = \beta D_H + (1-\beta) D_L$. Thus the dollar bid ask spread is $A - B = (1-\beta)(D_H - D_L)$.*

The endogenous transaction cost $(1-\beta)(D_H - D_L)$ captures the liquidity of the secondary market for corporate bonds. Later we will calculate the percentage bid-ask spread as the dollar spread divided by the mid point of transaction prices (bid price B and ask price A).

2.4 Summary of Setup

The model setup is summarized in a schematic representation given in Figure 1, and for exposition purposes we omit including the bankruptcy decision that are driven by the stochastic process in δ.

Primary market. Let us start with the firm. It (re)issues debt for D_H on the primary market to H/S types, as represented via the "Reissue" arrow. After the H/S-types buy the debt, there is a chance the bond matures before either a bankruptcy occurs or a liquidity shock hits. In this case, the bond goes back to the firm, which pays back the principal to the agent. This event is summarized in the "Maturity" arrow. This subpart of the graph represents the LT96 model. With liquidity shocks, an H/S-type transitions to an L-type with intensity ξ who values the bond at D_L, as represented by the "Liq. shock" arrow. Absent bankruptcy and retrading opportunities, the bond matures and the L-type will be paid back the face-value of the bond, again summarized by the "Maturity" arrow. $1/T$ indicates the flow of bonds that mature.

Secondary market. Once we introduce a secondary market, L-types can now sell the bond to H/S-types via the help of dealers. To do so, they contact dealers with an intensity λ, as indicated by the "Intermediation" arrow. They sell their bond to the dealer for $B = \beta(D_H - D_L) + D_L$. The dealer turns around and immediately (re)sells the bond on the inter-dealer market for $M = D_H$ to a dealer currently in contact with an H/S buyer, who in turn then sells the bond to an H/S-types, as indicated by the "Resale" arrow. It is important that H-types and not S-types are the marginal

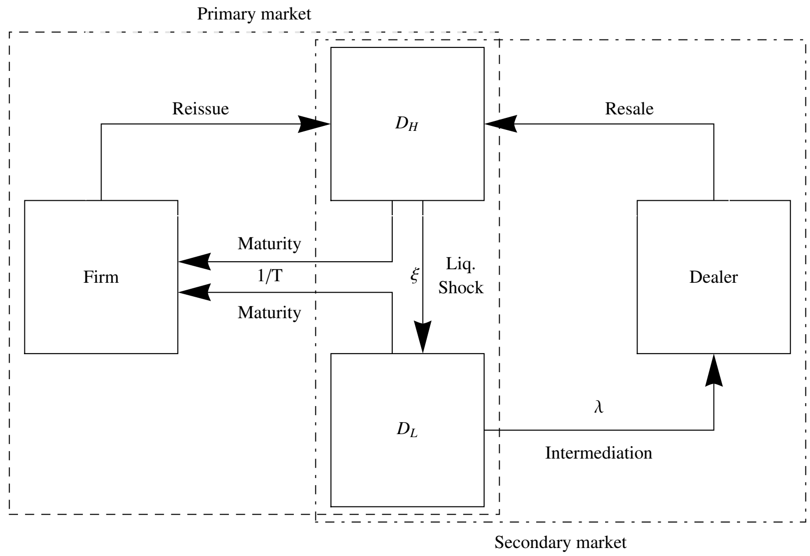

Figure 1: Schematic representation of model

agent for pricing on the inter-dealer market. Consequently, before the firm defaults, H-types are indifferent between staying out of the market, buying bonds of maturity T at reissue via the primary market, or buying bonds of maturities $\tau \in (0, T)$ on the secondary market. After default, S (L) types are buyers (sellers) of the defaulted bonds, and H types are waiting for bankruptcy payout.

As mentioned, we do not need to solve for S-type valuations to derive the pre-default equilibrium.[18] These value functions only matter for the dealer surplus appropriated, as the firm still issues at D_H in the primary market and the secondary market outcome for the L type sellers is only affected by D_H and D_L.

3 Model Solutions

3.1 Debt Valuations and Credit Spread

We first derive bond valuations by taking the firm's default boundary δ_b as given. Recall that $D_H(\delta, \tau)$ and $D_L(\delta, \tau)$ are the value of one unit of bond with time-to-maturity $\tau \leq T$, an annual

[18] S-type valuations would be a bit harder to solve, as we have to track S-types that currently do not hold the bond and their decision to buy (or not buy) the bond. This decision is complicated due to the holding restriction – to the extreme, if they buy a bond before the firm defaults, they cannot buy the defaulted bonds if the firm defaults later. We leave this issue for future research.

coupon payment of c, and a principal value of p to H and L investors, respectively. We have the following system of PDEs for the values of D_H and D_L, where we omit the two-dimensional argument (δ, τ) for both debt value functions:

$$\begin{aligned} rD_H &= c - \frac{\partial D_H}{\partial \tau} + \mu\delta \cdot \frac{\partial D_H}{\partial \delta} + \frac{\sigma^2\delta^2}{2}\frac{\partial^2 D_H}{\partial \delta^2} + \underbrace{\xi\left[D_L - D_H\right]}_{\text{Liqidity shock}}, \\ rD_L &= (c-\chi) - \frac{\partial D_L}{\partial \tau} + \mu\delta \cdot \frac{\partial D_L}{\partial \delta} + \frac{\sigma^2\delta^2}{2}\frac{\partial^2 D_L}{\partial \delta^2} + \underbrace{\lambda\left[B - D_L\right]}_{\text{Secondary market}} . \end{aligned} \tag{7}$$

The boundary conditions are $D_H = D_L = p$ at $\tau = 0$ because of the principal payment at maturity, and $D_i = \alpha_i \frac{\delta_b}{r-\mu}$ at $\delta = \delta_b$ where $i \in \{H, L\}$ as discussed in Section 2.3.1.[19]

The first equation in (7) is the type H bond valuation. The left-hand side rD_H is the required (dollar) return from holding the bond for type H investors. There are four terms on the right-hand side, capturing expected returns from holding the bond. The first term is the coupon payment. The next three terms capture the expected value change due to change in time-to-maturity τ (the second term) and fluctuation in the firm's fundamental δ_t (the third and fourth terms). The last term is a loss $D_L - D_H$ caused by the liquidity shock that transforms H investors into L investors, multiplied by the intensity of the liquidity shock.

The second equation in (7), the type L bond valuation, follows a similar explanation to the one above. The two differences are that there is a holding cost χ on the right hand side, and there is the value impact of the secondary market reflected in the last term of the right hand side. A type L investor meets a dealer with an intensity of λ and is then able to sell his bond (with a private value D_L) at a price of $B = (1-\beta) D_L + \beta D_H$. Plugging into equation (7), we have $\lambda (B - D_L) = \lambda\beta (D_H - D_L)$. One can thus interpret $\lambda\beta$ as the bargaining weighted intensity of "transitioning" (via a sale) back from the L state to the H state.[20] It is easy to show that when

[19] And, given any time-to-maturity τ, when $\delta \to \infty$, D_H and D_L converge to the values of default-free bonds (but still subject to liquidity shocks and search frictions).

[20] Although the debt values are functions of only the product $\beta\lambda$, the bid-ask spread is given by $(1-\beta)(D_H - D_L)$, and thus has β entering on its own independent of the product $\beta\lambda$. This will be important when calibrating our model, as it allows a separation of the identification of λ and β.

$\lambda \to \infty$, debt values converge to the LT96 case with perfectly liquid secondary markets. The surplus from intermediating trades vanishes because the outside option of meeting another dealer becomes very large.

We now define the matrix $\mathbf{A}$ that incorporates the discount factors and the effective transition intensities ξ and $\lambda\beta$ of the states. Then, the following decomposition holds:

$$\mathbf{A} \equiv \begin{bmatrix} r+\xi & -\xi \\ -\lambda\beta & r+\lambda\beta \end{bmatrix} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}.$$

where $\hat{\mathbf{D}} \equiv \operatorname{diag}[\hat{r}_1, \hat{r}_2]$, $\hat{r}_1 = r+\xi+\lambda\beta > r = \hat{r}_2$, is the matrix of eigenvalues of $\mathbf{A}$, and $\mathbf{P}$ is the matrix of stacked eigenvectors. For a given default boundary δ_b, we derive the closed-form solution for the bond values in the next proposition.[21]

Proposition 4 *The debt values are given by*

$$\begin{bmatrix} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{bmatrix} = \mathbf{P}\left[\hat{\mathbf{k}}_0^D + \exp\left(-\hat{\mathbf{D}}\tau\right)\hat{\mathbf{k}}_F^D\left[1-F(\delta,\tau)\right] + \mathbf{G}(\delta,\tau)\cdot\hat{\mathbf{k}}_G^D\right]. \tag{8}$$

Here, by defining $a \equiv \frac{\mu}{\sigma^2} - 0.5$, $\varphi_1 \equiv 0$, $\varphi_2 \equiv -2a$, $\gamma_{j1,2} \equiv -a \pm \sqrt{a^2 + \frac{2}{\sigma^2}\hat{r}_j}$, $\mathbf{c} \equiv [c, c-\chi]^\top$, *and* $q(\delta,\rho,t) \equiv \frac{\log(\delta_b) - \log(\delta) - (\rho+a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$, *the constants in (8) are given by:*

$$\hat{\mathbf{k}}_0^D \equiv \hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{c},\ \ \hat{\mathbf{k}}_F^D \equiv p\mathbf{P}^{-1}\mathbf{1} - \hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{c},\ \ \hat{\mathbf{k}}_G^D \equiv \mathbf{P}^{-1}\boldsymbol{\alpha}\frac{\delta_b}{r-\mu} - \hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}\mathbf{c}\ ,$$

and the functions are given by $\mathbf{G}(\delta,\tau) = \begin{bmatrix} G_1(\delta,\tau) & 0 \\ 0 & G_2(\delta,\tau) \end{bmatrix}$,

$$F(\delta,\tau) \equiv \sum_{i=1}^{2}\left(\frac{\delta}{\delta_b}\right)^{\varphi_i} N\left[q(\delta,\varphi_i,\tau)\right],\ G_j(\delta,\tau) \equiv \sum_{i=1}^{2}\left(\frac{\delta}{\delta_b}\right)^{\gamma_{ji}} N\left[q(\delta,\gamma_{ji},\tau)\right],$$

[21] All derivations, because of the linear decomposition, would go through if the capital structure of the firm consisted of different issues of debt differing in T.

where $N(x)$ is the cumulative distribution function for a standard normal distribution.

A closer inspection of the solution reveals a linear combination (via the matrix $\mathbf{P}$) of two sub-solutions each closely related to the original LT96 solution: the first term gives the value of a risk-free consol bond, the second term encapsulates the probability that the bond will mature before default, and the third term encapsulates the probability that the bond will default before maturity. Relative to LT96, each of these independent sub-solutions $i = \{1, 2\}$ has a distorted discount rate $\hat{r}_i$, a distorted coupon rate $\hat{c}_i \equiv \left(\mathbf{P}^{-1}\mathbf{c}\right)_i$ and a distorted recovery value $\hat{\alpha}_i \equiv \left(\mathbf{P}^{-1}\boldsymbol{\alpha}\right)_i$.[22]

Credit Spreads. Recall that the bond credit spread is the spread between the corporate bond yield and the risk-free rate r. Given a bond of value $D(\delta, \tau)$, the bond yield is defined as the solution to the following equation:

$$D(\delta,\tau) = \frac{c}{yield}(1 - e^{-yield\cdot\tau}) + pe^{-yield\cdot\tau}, \tag{9}$$

so that the right-hand side is the present value of a bond (discounted by *yield*) with a constant coupon payment c and a principal payment p, conditional on it being held to maturity without default or re-trading. For the remainder of the paper, we simply use the ask price $D_H(\delta, \tau)$ in Proposition 4 as our bond price for the left-hand side of equation (9).

3.2 Equity Valuation and Firm Value

The next key step is the equity holders' decision to default, given that they receive the net cash flow in (2) every instant. Because equity is naturally an infinite maturity security and we are investigating a stationary (debt maturity structure) setting, the equity value $E(\delta; \delta_b)$ satisfies the following ordinary differential equation:

$$rE = \delta - (1-\pi)c + \underbrace{\frac{1}{T}\left[(1-\kappa)D_H(\delta, T) - p\right]}_{\text{Rollover}} + \mu\delta E' + \frac{\sigma^2\delta^2}{2}E'', \tag{10}$$

[22]Given a matrix $\mathbf{M}$, $(\mathbf{M})_i$ selects the i-th row and $(\mathbf{M})_{ij}$ selects the i-th row and j-th column.

where the left hand side is the required rate of return of equity holders. On the right hand side, the first three terms are the equity holders net cash flows, and the next two terms are capturing the instantaneous change of the firm fundamental. As mentioned earlier, the term involving square brackets is the cash-flow term that arises from rolling over debt (while keeping coupon, principal, and maturity stationary), with $1/T$ being the rollover frequency.

It is worthwhile to point out that equity value in our model is no longer the difference between the levered firm value and debt value adjusted for tax benefits and bankruptcy costs, a common calculation performed in Leland-type models. This is because part of the firm value goes to the dealers in the secondary bond market, and part vanishes because of inefficient holdings of bonds by L types. Instead, we need to solve for $E(\delta)$ directly via (10), which is non-trivial due to the highly-nonlinear form of $D_H(\delta, T)$ given in (8). The next proposition gives the equity value.

Proposition 5 *Given a default boundary δ_b, the equity value is given by*

$$E(\delta;\delta_b) = k_2^E \left(\frac{\delta}{\delta_b}\right)^{\eta_2} + \frac{\delta}{r-\mu} + k_0^E + \frac{1}{T}(1-\kappa)\,\mathbf{S}\cdot\mathbf{P}\left[-\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F(\delta) + \mathbf{g}_G(\delta)\,\hat{\mathbf{k}}_G^D\right], \quad (11)$$

where $\mathbf{g}_G(\delta) = \begin{bmatrix} g_{G_1}(\delta) & 0 \\ 0 & g_{G_2}(\delta) \end{bmatrix}$, $\eta_{1,2} \equiv -a \pm \sqrt{a^2 + \frac{2}{\sigma^2} r}$, $\Delta\eta \equiv \eta_1 - \eta_2$, *and*

$$\begin{aligned}
k_0^E &\equiv \tfrac{1}{r}\left\{-(1-\pi)c + \tfrac{1}{T}\left[(1-\kappa)\,\mathbf{S}\cdot\mathbf{P}\left(\hat{\mathbf{k}}_0^D + \exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D\right) - p\right]\right\}, \\
k_U^E &\equiv -\left(\tfrac{\delta_b}{r-\mu} + k_0^E + \tfrac{1}{T}(1-\kappa)\,\mathbf{S}\cdot\mathbf{P}\left[-\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F(\delta_b) + \mathbf{g}_G(\delta_b)\,\hat{\mathbf{k}}_G^D\right]\right), \\
g_F(x) &\equiv \tfrac{1}{-\Delta\eta}\tfrac{2}{\sigma^2}\textstyle\sum_{i=1}^2\left\{\tfrac{x^{\eta_2}}{\delta_b}H(x,\varphi_i,\eta_2,T) - \tfrac{x^{\eta_1}}{\delta_b}H(x,\varphi_i,\eta_1,T)\right\}, \\
g_{G_j}(x) &\equiv \tfrac{1}{-\Delta\eta}\tfrac{2}{\sigma^2}\textstyle\sum_{i=1}^2\left\{\tfrac{x^{\eta_2}}{\delta_b}H(x,\gamma_{ji},\eta_2,T) - \tfrac{x^{\eta_1}}{\delta_b}H(x,\gamma_{ji},\eta_1,T)\right\}, \\
H(\delta,\rho,\eta,T) &\equiv \begin{cases} \tfrac{1}{\eta-\rho}\left\{\delta^{\rho-\eta}N\left[q(\delta,\rho,T)\right] - \delta_b e^{\frac{1}{2}\left[(\eta+a)^2-(\rho+a)^2\right]\sigma^2 T}N\left[q(\delta,\eta,T)\right]\right\} & \rho\neq\eta \\ \sigma\sqrt{T}\left[q(\delta,\rho,T)\,N\left[q(\delta,\rho,T)\right] + \phi\left(q(\delta,\rho,T)\right)\right] & \rho=\eta \end{cases},
\end{aligned}$$

where $q(\cdot,\cdot,\cdot)$ is given in Proposition 4 and $\phi(x)$ is the marginal distribution function for a standard Normal distribution.

Following LT96, we assume that at time 0 the firm issues new bonds at a price $D_H(\delta_0, \tau)$, where maturities are uniformly distributed $\tau \in [0, T]$, on the primary market. Given the results established above, the levered initial firm value $TV_0(\delta_0, T; \delta_b)$ is the sum of equity valuation plus how much money has been raised by the bond issuance[23]:

$$
\begin{aligned}
TV_0(\delta_0, T; \delta_b) &= E(\delta_0; \delta_b) + (1-\kappa)\frac{1}{T}\int_0^T \mathbf{S}\cdot\mathbf{D}(\delta_0, \tau; \delta_b)\, d\tau \\
&= E(\delta_0) \qquad (12) \\
&\quad + (1-\kappa)\,\mathbf{S}\cdot\mathbf{P}\left[\hat{\mathbf{k}}_0^D + \frac{1}{T}\hat{\mathbf{D}}^{-1}\left(\mathbf{I} - \mathbf{G}(\delta_0, T) - \exp\left(-\hat{\mathbf{D}}T\right)[1 - F(\delta_0, T)]\right)\hat{\mathbf{k}}_F^D + \mathbf{J}(\delta_0, T)\,\hat{\mathbf{k}}_G^D\right]
\end{aligned}
$$

where $\mathbf{J}(\delta, T) = \begin{bmatrix} J_1(\delta, T) & 0 \\ 0 & J_2(\delta, T) \end{bmatrix}$, $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and

$$
J_j(\delta, T) = \frac{1}{(\gamma_{1j} + a)\,\sigma\sqrt{T}} \sum_{i=1}^{2} (-1)^i \left(\frac{\delta}{\delta_b}\right)^{\gamma_{ji}} N\left[q(\delta, \gamma_{ji}, T)\right] q(\delta, \gamma_{ji}, T).
$$

We will use this measure in section 5.1 to study the optimal maturity structure decision by the firm at time 0.

3.3 Endogenous Default Boundary

So far we have taken the default boundary δ_b as given. We now use the standard smooth pasting condition $E_\delta(\delta_b; \delta_b) = 0$ to determine the optimal δ_b chosen by equity holders in closed form.

Proposition 6 *The endogenous default boundary δ_b is given by*

$$
\begin{aligned}
\delta_b(T) &= (r-\mu)\left[\eta_2 - 1 + \frac{1}{T}(1-\kappa)\,\mathbf{S}\cdot\mathbf{P}\cdot\mathbf{h}_G\cdot\hat{\boldsymbol{\alpha}}\right]^{-1} \\
&\quad \times \left[-\eta_2 k_0^E + \frac{1}{T}(1-\kappa)\,\mathbf{S}\cdot\mathbf{P}\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D h_F + \frac{1}{T}(1-\kappa)\,\mathbf{S}\cdot\mathbf{P}\cdot\mathbf{h}_G\cdot\hat{\mathbf{k}}_0^D\right],
\end{aligned}
$$

[23]The reader should note the difference that we have one unit measure of bonds, whereas LT96 expand the measure of bonds according to maturity.

where $\hat{\boldsymbol{\alpha}}_i \equiv \mathbf{P}^{-1}\boldsymbol{\alpha}_i,\ \mathbf{h}_G = \begin{bmatrix} h_{G_1} & 0 \\ 0 & h_{G_2} \end{bmatrix}$, *and*

$$\begin{aligned} h_F &\equiv -\frac{2}{\sigma^2}\sum_{i=1}^{2}\frac{1}{\eta_1-\varphi_i}\left\{N\left[-(\varphi_i+a)\,\sigma\sqrt{T}\right]-e^{rT}N\left[-(\eta_1+a)\,\sigma\sqrt{T}\right]\right\}, \\ h_{G_j} &\equiv \begin{cases} -\frac{2}{\sigma^2}\sum_{i=1}^{2}\frac{1}{\eta_1-\gamma_{ji}}\left\{N\left[-(\gamma_{ji}+a)\,\sigma\sqrt{T}\right]-e^{(r-\hat{r}_j)T}N\left[-(\eta_1+a)\,\sigma\sqrt{T}\right]\right\} & \gamma_{ji}\neq\eta_1 \\ \textit{see Appendix} & \gamma_{ji}=\eta_1 \end{cases} \end{aligned}$$

Relating to existing literature, in the absence of debt rollover, secondary market frictions cannot affect the equity holders' default decision once debt is in place. Infinite debt maturity features no rollover and thus no feedback between liquidity and default, and thus the model converges to the bankruptcy boundary derived in Leland (1994).

3.4 Endogenous Liquidity

In our model, the (dollar) bid-ask spread is the difference between the bid price $B(\delta,\tau)$ and the ask price $A(\delta,\tau) = D_H(\delta,\tau)$, which is a constant positive fraction of the surplus $\Pi(\delta,\tau)$:

$$A(\delta,\tau) - B(\delta,\tau) = (1-\beta)\,\Pi(\delta,\tau) = (1-\beta)\left[D_H(\delta,\tau) - D_L(\delta,\tau)\right]. \tag{13}$$

Empirically, the effective percentage bid-ask spread $\Delta(\delta,\tau)$ is more commonly used, which is often define as the dollar bid-ask spread divided by the mid point of transaction prices:

$$\Delta(\delta,\tau) = \frac{A(\delta,\tau)-B(\delta,\tau)}{\frac{1}{2}A(\delta,\tau)+\frac{1}{2}B(\delta,\tau)} = \frac{2(1-\beta)\left[D_H(\delta,\tau)-D_L(\delta,\tau)\right]}{(1+\beta)\,D_H(\delta,\tau)+(1-\beta)\,D_L(\delta,\tau)}. \tag{14}$$

Fixing cash flows δ, we can show the bid-ask spread is increasing in time-to-maturity τ. As in Feldhutter (2011), this is because a shorter time-to-maturity delivers the full principal back to L type investors sooner, and this enhances L type's outside option in the bargaining and reduces the rent extracted by dealers, thereby resulting in a smaller bid-ask spread. In fact, by the boundary conditions the surplus vanishes as time-to-maturity goes towards 0, i.e., $\lim_{\tau\to 0}\Pi(\delta,\tau) = 0$. Intuitively, if the bond is almost immediately demandable from the firm, L type investors gain little value from trade with dealers, and as a result the bid-ask spread vanishes. This indicates that short-term debt provides liquidity for bond investors, and we will discuss the role of liquidity provision in more detail in Section 5.1.

We focus more on the effect of firm fundamental δ on $\Delta(\delta,\tau)$, as our goal is to develop a structural model that can generate the empirical regularity of higher transaction costs for bonds issued by lower rating firms. When $\delta = \infty$, so that bonds are default-free, we have

$$\begin{bmatrix} D_H(\infty,\tau) \\ D_L(\infty,\tau) \end{bmatrix} = \mathbf{A}^{-1}\mathbf{c} + \exp(-\mathbf{A}\tau)\left(\mathbf{p} - \mathbf{A}^{-1}\mathbf{c}\right)$$

which gives a strictly positive bid-ask spread for default-free bonds. In order for the bid-ask spread to be decreasing in distance-to-default, one necessary condition is that the bid-ask spread at $\delta \downarrow \delta_b$ is higher than that of default-free bonds. Throughout the paper, we focus on this situation, which mandates that the valuation wedge between H and L investors when the firm defaults, i.e. $\alpha_H - \alpha_L = \frac{\chi_b}{r+\theta+\xi_b} > 0$, is sufficiently higher than the valuation wedge between H and L investors when the firm far away from bankruptcy. In our model, there are several economic forces that drive a greater valuation wedge $D_H - D_L$ when the firm is in default. First, the secondary market of non-defaulted bonds is a *seller's market* (i.e., more buyers from H/S investors than sellers from L investors), but once the firm defaults, there are few potential buyers of defaulted securities (only S investors). As a result, the secondary market of defaulted bonds turns into a *buyer's market* (more L type sellers than S type buyers), and L types lose all their potential trading surplus. Second, L type sellers will face a greater holding cost for defaulted bonds, which drives D_L valuation even further.

In the Appendix, we provide sufficient conditions in Proposition 7 that guarantee that the dollar bid-ask spread $\Pi(\delta,\tau)$ decreases with δ, which also guarantees that the percentage illiquidity

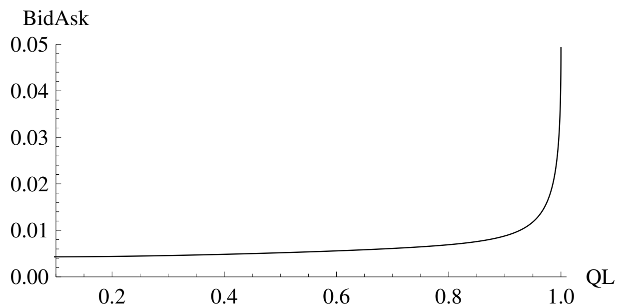

Figure 2: **Proportional bid-ask spread Δ w.r.t.** QL**:** Varying δ_0 while $c = 390bps$.

$\Delta(\delta, \tau)$ decreases with δ.[24]

We plot the bid-ask spread in Figure 2 as a function of firrm's distance-to-default. To be consistent with empirical moments, the distance-to-default is measured by firm's quasi-market leverage, i.e., the ratio of the book value of debt (which is p) to the sum of book value of debt and market value of equity:

$$QL(\delta; T) = \frac{p}{E(\delta; T) + p}.$$

Note that $QL \in [0, 1]$ with $QL(\delta_b) = 1$, $\lim_{\delta \to \infty} QL(\delta) = 0$, and $QL'(\delta) < 0$. That the bid-ask spread is decreasing with distance-to-default is consistent with the empirical regularities found in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011).

3.5 Positive Feedback between Default and Liquidity

By linking the secondary market liquidity endogenously to firm fundamental, we now demonstrate the positive default-liquidity spiral in which the deterioration of firm fundamental, via worsening liquidity of the secondary bond market, edges the firm even closer to default, which in turn leads to further deterioration in secondary market liquidity.

[24]One can show that $\Pi_\delta < 0$ implies $\Delta_\delta < 0$. To see this, note that $\Delta_\delta = \frac{2(1-\beta)(\partial_\delta D_H \cdot D_L - \partial_\delta D_L \cdot D_H)}{(\cdot)^2}$. Using $\Pi_\delta < 0 \iff \partial_\delta D_H < \partial_\delta D_L$, we have $(\partial_\delta D_H \cdot D_L - \partial_\delta D_L \cdot D_H) < \partial_\delta D_L (D_L - D_H) = -\partial_\delta D_L \Pi < 0$ and thus $\Delta_\delta < 0$.

3.5.1 Rollover Losses, Endogenous Liquidity, and Endogenous Default

To understand the mechanism, consider the rollover losses borne by equity holders as a function of the firm cash flow rate δ, which enters the net cash flows that equity holders are receiving (recall Eq. (2)):

$$NC_t = \underbrace{\delta_t}_{\text{CF}} - \underbrace{(1-\pi)\,c}_{\text{Coupon}} + \underbrace{\frac{1}{T}\left[(1-\kappa)\,D_H\left(\delta_t, T;\ illiquidity\right) - p\right]}_{\text{Rollover}}. \tag{15}$$

Without secondary market illiquidity (say LT96), when the firm fundamental δ deteriorates, the (absolute value) of rollover losses $\frac{1}{T}\left[D\left(\delta, T\right) - p\right]$ rises because investors adjust the market price of newly issued bonds downward.

When the secondary market for corporate bonds are illiquid, "illiquidity" of the secondary market enters (15) in the bond pricing $D_H\left(\delta_t, T;\ illiquidity\right)$, as H type investors who purchase bonds on the primary market worry about the illiquidity they will face once hit by a liquidity shock and thus forced to sell in the secondary market. When secondary market illiquidity is high, equity holders have to absorb greater rollover losses. This lowers the equity holders' option value of keeping the firm alive by servicing the debt, leading to earlier default.

In models with constant secondary market liquidity (e.g., He and Xiong (2012b)), "illiquidity" enters $D_H\left(\delta_t, T;\ illiquidity\right)$ in (15) exogenously, and does not depend on the firm's distance-to-default. In contrast, in our model "illiquidity" in $D_H\left(\delta_t, T;\ illiquidity\right)$ also depends firm fundamental δ. More importantly, the secondary market liquidity is pro-cyclical, i.e., liquidity worsens and thus depresses the primary market issuance price $D_H\left(\delta, T\right)$ further for firms that are closer to default. This reduces the equity holders' option value of servicing the debt *especially in bad times*, and hence the firm defaults even earlier .

3.5.2 Positive feedback between default and liquidity

The above discussion implies an important positive feedback loop between firm default and secondary market liquidity for corporate bonds, which is illustrated in Figure 3. Imagine a *negative* shock to the firm's cash flow rate δ. Since this negative shock brings the firm closer to default,

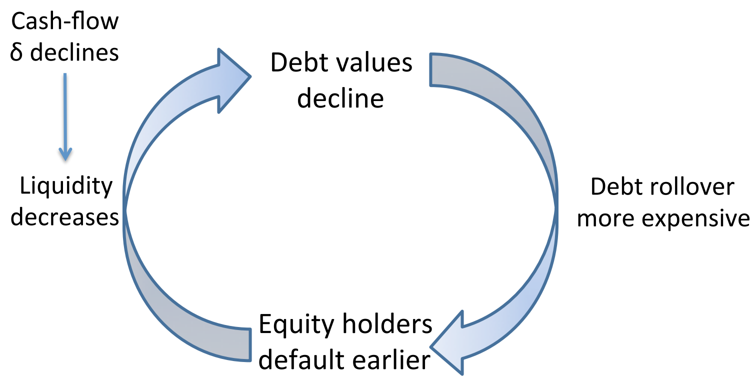

Figure 3: **Feedback loop** between secondary market liquidity and equity holders' default decision.

this constitutes a pure-fundamental driven negative shock to bond investors and lowers the holding values of D_H and D_L. This force is already present in LT96 and He and Xiong (2012b).

The novelty of our model is that a negative δ shock not only lowers debt values, but also worsens the secondary market liquidity. The lower distance to default worsens the L types' outside option when bargaining with a dealer, not only because default will turn the secondary market from a seller's market to a buyer's market, but also because default will lead to protracted bankruptcy court decisions and prohibitive holding costs. Consequently, even before default, bonds in the secondary market become more illiquid, as indicated by the left large arrow with "declining liquidity" in Figure 3. This is the pro-cyclicality of liquidity we already discussed above.

Rational H type bond investors will thus value bonds less, i.e., a lower D_H, because they expect to face a less liquid secondary market once hit by liquidity shocks. As shown in Figure 3, the worsening liquidity in the secondary market gives rise to a lower primary market bond issuing price D_H relative to an environment with constant market liquidity.

The lower bond prices now feed back to the equity holders' default decision via the rollover channel, indicated by the arrow on the right of Figure 3. This is because equity holders are absorbing heavier rollover losses (i.e. net cash flow NC_t in (15) goes down). Equity holders hence default earlier at a higher threshold δ_B. The higher default threshold now translates into a shorter distance to default $\delta - \delta_B$. But just as discussed before, the search-based secondary market kicks in again: as shown on the left-hand side in Figure 3, the shorter distance to default *further* worsens market liquidity via the declining outside option of the L type investors. The outcome of this thought experiment is the fixed point δ_b given in Proposition 6.

4 Calibration

We calibrate our model to explore the model's cross-sectional implications in this section. We first explain our parameter choices in Section 4.1. Section 4.2 presents the calibration of the model to bonds of different rating classes, and Section 4.3 discusses the ability of the model to decompose the credit spread into default and liquidity components.

4.1 Parameters

4.1.1 Parameters for Search Frictions

We rely on implied bond illiquidity to determine parameters on search frictions. We choose the liquidity shock intensity $\xi = 1$ which implies that investors are hit by liquidity shocks once every year; this roughly matches the empirical turnover of 1 year for corporate bonds (see Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011)).[25] As explained later, we choose the holding costs parameters $\chi_0 = 0.55\%$ and $\chi_1 = 1/3$ to target the bid-ask spread of around 50 *bps* for BBB rated bonds (see Section 4.2).

There are two other parameters, β and λ, that are relatively difficult to set due to lack of empirical evidence. For the bargaining power allocation between dealers and investors, we set $\beta = 15\%$ (i.e., dealers get 85% of the trading surplus) in the baseline case. To the best of our knowledge, Feldhutter (2011) is the only paper that provides an estimation of $\beta = 3\%$ based on a different structural model, which suggests a much lower investors' bargaining power. Presumably, with typical corporate bond investors being sophisticated institutions, one would expect a higher investors' bargaining power.

[25] According to Bao, Pan, and Wang (2011), the average turnover in their sample is about 1 year. In our model, the average time that an investor is holding the bond (including the time that the investor remains at H type and that he is L type but searching) is $\frac{1}{\xi} + \frac{1}{\lambda}$. As we will set $\lambda = 15$ in the baseline, $\frac{1}{\xi} + \frac{1}{\lambda}$ is about 1.06.

The second parameter λ is the meeting intensity with dealers. Although we are using search-based framework to model secondary corporate bond market, we would like to interpret the trading friction in our model more broadly. For instance, the average time spent during search, which is $1/\lambda$ in the model, can be interpreted as the time it takes for the investors who are hit by liquidity shocks to sell their holding completely.[26] We choose $\lambda = 15$ in the baseline, so that it takes about 3.5 weeks to sell their bond holdings completely.[27]

4.1.2 Effective Recovery Rates at Default

The effective recovery rates at default, i.e., α_H and α_L are the only two parameters that anchor the pre-default prices. In our fully developed structural model, the effective recovery rates at default for H and L investors are an endogenous outcome from the search-based secondary market for defaulted bonds. Unfortunately, there is very little empirical data to inform us about those deeper parameters that govern the post-default market illiquidity.

To address this problem, for the purpose of calibration, as in Cui, Chen, He, and Milbradt (2013), we will use the buy-and-hold returns of defaulted bonds from the Moody's default and recovery database to identify the effective recovery rates at default. For defaulted corporate bond between 1987 and 2011, this dataset gives the trading price right after default and its eventual recovery value at the settlement (or emergence) date.[28] To account for systematic risk in the buy-and-hold returns in investing the defaulted bonds, we borrow from VC/PE literature to adjust for risk by discounting the return for each defaulted bond by a public market reference return over the holding horizon, where the market reference is simply S&P500 total return (including dividends).[29]

[26]This includes the time that the investor needs to find the right dealer(s) who have either the right inventories or the right trading partners, as well as the time that this dealer needs to find the right trading partners. We are unaware of any empirical studies on this topic.

[27]Although our model does not feature aggregate states, the choice of λ should represent the weighted average of searching length over business cycle under risk-neutral measure; if it is much harder to find dealers to complete the trade in bad times, then the lengthy waiting time in bad times should receive a greater weight.

[28]The trading price right after default is the first transaction price within 3 months after default. The eventual recovery value is the so-called emergence price, which can be either trading price, settlement price, or liquidation price at the time of bankruptcy emergence. We follow the Moody prefered choice in deciding the bond's eventual recovery value.

[29]We borrow this approach from the VC/PE literature because it is difficult to estimate *beta* for this investment strategy due to unbalanced panels and unknown interim returns before emergence date, a well-known problem in the

In the Moody's default and recovery database, the average time to ultimate resolution is 501 days (so 1.37 years), leading us to set $\theta = 0.73$. The annualized excess return for a buy-and-hold strategy for defaulted bonds is 36%, implying an expected total holding return of around 52% over 1.37 years.

To convert these numbers to the estimates of α_L and α_H, we follow an approach common in structural corporate bond pricing models, say Chen, 2011; Bhamra, Kuehn, and Strebulaev, 2010, who take the trading price right after default as the bankruptcy recovery $\alpha\frac{\delta_b}{r-\mu}$. The estimate of α from Chen (2011) is about 0.5 (average across aggregate states), implying the trading price right after default is $0.5\frac{\delta_b}{r-\mu}$. Because the trading price is likely to be the bid price which is $\alpha_L\frac{\delta_b}{r-\mu}$ in our model, we thus set $\alpha_L = 0.5$. The buy-and-hold return implies that the eventual recovery in our model, as a fraction of unlevered firm value, is $\alpha = \alpha_L 1.52 = 0.75$.[30]

Finally, we rely on the H type valuation equation (normalized by $\frac{\delta_b}{r-\mu}$) $r\alpha_H = \theta(\alpha - \alpha_H) + \xi_b(\alpha_L - \alpha_H)$ in (3) to pin down α_H. There is one free parameter ξ_b, which is the H investors' liquidity shock intensity after default, to be determined. We choose $\xi_b = 5$ so that $\alpha_H = 0.53$, and the implied bid-ask spread right before default is about 5% (as $\alpha_L = 0.5$). Relative to the pre-default shock intensity $\xi = 1$, an H investor holding a defaulted bond is 5 times more likely to be hit by a liquidity shock than an S investor holding a defaulted bond.[31]

4.1.3 Cash flows parameters

Without loss of generality, we normalize face value to $p = 1$. We set the risk-free rate $r = 2\%$ which is commonly used in the macroeconomics literature to match the real interest rate. The cash flow rate volatility of $\sigma = 25\%$ is also standard in the structural debt literature. We set the cash flow drift under $\mathcal{Q}$ to $\mu = 0$, which essentially affects the overall match between credit spreads and

VC/PE literature. And, based on the unbalanced panel we find that the market *beta* of this strategy is about 0.7, which potentially would make the excess return even higher.

[30] This is broadly consistent with Andrade and Kaplan (1998) who find the cost of financial distress is about 10-20% of firm value.

[31] Note that a higher ξ_b leads to a lower wedge $\alpha_H - \alpha_L$. We have chosen a reasonably large increase in the liquidity shock intensity post-default which is likly to be an upper end of the estimate for the class of investors who invested in distressed but yet-to-default bonds.

Firm Characteristics			Illiquid Secondary Market		
Parameter	Interpretation	Value	Parameter	Interpretation	Value
σ	Volatility	25%	r	Interest rate rate	2%
μ	Drift	0%	χ_0	Holding cost prop to p	0.55%
π	Tax shield	27%	χ_1	Holding cost prop to c	$1/3$
p	Principal	1	ξ	Intensity of liquidity shock	1
c	Coupon	3.9%	λ	Intensity to meet dealers	15
T	Bond maturity	10	β	Bargaining power of investors	15%
κ	Issuance costs	1%	α_H	Recovery value H type	53%
δ_0	Initial Cash Flow	0.0547	α_L	Recovery value L type	50%

Table 1: **Model parameters** for baseline calibrations.

leverage.[32] We also use a debt tax benefit rate of $\pi = 27\%$ to take into account the effect that many corporate bond investors are tax-exempt financial institutions.[33] We set debt maturity of $T = 10$ years, as most of the calibration exercises on corporate bond pricing focus on the maturity of 10-years. As T also captures the overall debt maturity structure of the firm, this choice implies that both the mean and median of maturity of the firm's outstanding debt are $\frac{T}{2} = 5$ years, roughly consistent with Custodio, Ferreira, and Laureano (????).

We set the coupon c to target a credit spread of 190 *bps* for BBB rated bonds according to Huang and Huang (2003). The further requirement that the bond is issued at par pins down the initial cash-flow level $\delta_0 = 0.0547$.

4.2 Calibration for Bonds with Different Rating Groups

We investigate the quantitative performance of our model for corporate bonds across rating classes. There is an important reason for us to focus on cross-sectional predictions, as one of the main advantages of our corporate bond pricing model is that the secondary bond market liquidity endogenously depends on the firm's distance-to-default. Relative to the previous literature with exogenous secondary market liquidity (say He and Xiong (2012b)), tying secondary market liquid-

[32]Our model is cast in a risk-neutral world. This choice of $\mu = 0$ is consistent with a drift of 3% under the physical measure $\mathcal{P}$, a volatility of 10% on systemic risk, and a price of risk (or Sharp ratio) of 30%.

[33]While tax rate of bond income is 32%, many institutions holding corporate bonds enjoy tax exemption. Thus, we use an effective bond income tax rate of 25%. Then, the formula given by Miller (1977) implies a debt tax benefit of $1 - [(1 - 32\%)(1 - 15\%) / (1 - 25\%)] = 26.5\%$ where 32% is the marginal rate of corporate tax and 15% is the marginal rate of capital gain tax.

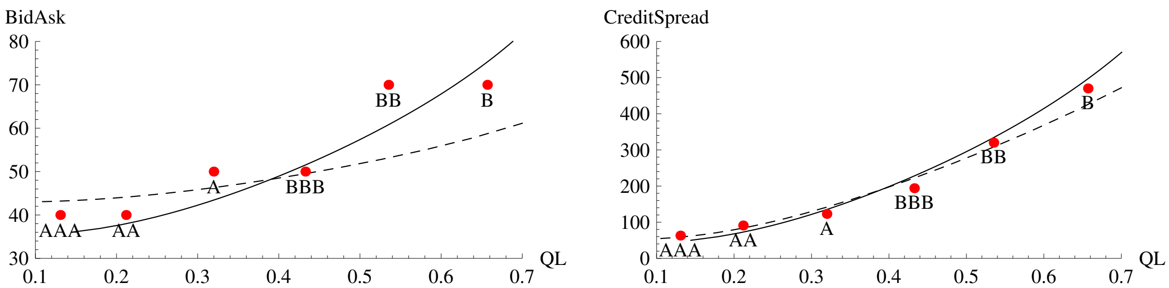

Figure 4: **Calibration results:** Left panel: Quasi Leverage vs Proportional Bid-Ask. Right panel: Quasi Leverage vs Credit spread. Solid line: Adjusting c so always priced at par. Dashed line: Fixed $c = 390bps$. Dots: Empirical data.

ity to firm's distance-to-default results in a parsimonious model as we can generate the empirical cross-sectional pattern of illiquidity across credit ratings by adjusting the firm's distance-to-default only. In that regard, the model imposes tighter restrictions on the calibration via this joint matching requirement.

For empirical moments, we take the leverages and credit spreads data for corporate bonds across six rating classes (from AAA to B) from Huang and Huang (2003). For bond liquidity information across rating classes, Edwards, Harris, and Piwowar (2007) study the bid-ask spreads for three categories: *superior* grade that covers ratings from AAA to AA, *investment* grade that covers ratings from A to BBB, and junk grade that covers ratings below BB. At the median trade size of \$240K, they document a bid-ask spread of $40bps$ for bonds with superior grade, a bid-ask spread of $50bps$ for investment grade, and a bid-ask spread of $70bps$ for junk grade.[34]

When we vary credit ratings, essentially we are varying a firm's leverage or distance-to-default. We calculate the model implied bid-ask spreads and credit spreads for two different kinds of bonds, depending on whether we adjust the coupon to ensure the bond is priced at par (recall that we always normalize the principal $p = 1$). For the first kind of bond, we only change the firm fundamental δ_0 without adjusting coupon, and thus away from $\delta_0 = 0.0547$ the bond is no longer priced at par. This treatment corresponds to bonds that have been issued in the past, and fluctuating firm fundamentals lead these bonds to receive different ratings (e.g., fallen angels). For the second kind

[34] This is taken from Edwards, Harris, and Piwowar (2007), Panel B in Figure 3, page 1441.

of bond, we always adjust the coupon c to ensure that the bond is priced at par at whatever the prevailing δ_0 is. This treatment corresponds to newly issued bonds for different firms, which is a standard practice in structural corporate bond pricing literature. In our model, because the holding costs in the liquidity state may depend on the relative size of coupon versus principal, these two treatments may give different results. In the data, bonds in each rating class can be either seasoned bonds with rating changes or newly issued bonds, and thus we present both calibrations.

The results are shown in Figure 4, with the left panel for bid-ask spreads and the right panel for credit spreads. On each panel, the empirical moments across the rating classes are plotted as solid dots. The solid line graphs the model implied moments for bonds with coupon adjustment, i.e., bonds that are always issued at par, while the dashed line graphs the model implied moments for bonds without coupon adjustment. In the right panel, we see that the model implied credit spreads match the cross-sectional empirical pattern quite well.

The left panel shows model implied bid-ask spreads as well as empirical moments across different ratings. We have chosen the holding costs parameters $\chi_0 = 0.55\%$ and $\chi_1 = 1/3$ so that $\chi = \chi_0 + \chi_1 c$ to target the bid-ask spread of around 50 *bps* for BBB rated bonds, and the bid-ask spread of around 40 *bps* for AA rated bonds. The left panel shows that the model implied bid-ask spread will depend on whether we adjust the coupon rate across different ratings. More specifically, for bonds with coupon adjustment, the implied bid-ask spreads tend to vary with credit ratings more than the data suggests; while for bonds without coupon adjustment, the implied bid-ask spreads tend to vary with credit ratings less than the data suggests. This is not surprising, as the holding costs are affected by the coupon level. Overall, Figure 4 suggests that our baseline model does a reasonably good job at jointly matching the cross-sectional pattern of credit spread and liquidity quantitatively.

4.3 Liquidity Premium and Default Premium

It has been widely recognized that the credit spread of corporate bonds not only reflects a default premium determined by the firm's credit risk, but also a liquidity premium due to the illiquidity of the secondary debt market, e.g., Longstaff, Mithal, and Neis (2005), and Chen, Lesmond, and

Wei (2007). However, both academics and policy makers tend to treat the default premium and liquidity premium as independent, and thus ignore important interactions between them. For instance, it is common practice to decompose firms' credit spreads into independent liquidity-premium and default-premium components and then assess their quantitative contributions, e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010).

This assumption of independence between liquidity and default is contrary to the data, which suggests that these components exhibit strong positive correlation. We have seen in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011) that liquidity deteriorates for bonds that are issued by firms with high CDS spreads. In a report issued by Barclay Capital, Dastidar and Phelps (2009) study the quote-based bond liquidity measure directly, and document the same robust empirical regularity in not only cross-section (investment grade vs. speculative grate) but also time-series (2005-06 before crisis vs 2008-09 during crisis). More recently, Dick-Nielsen, Feldhutter, and Lando (2012), and Friewald, Jankowitsch, and Subrahmanyam (2012) document that liquidity in corporate bond market dries up substantially during the 2007/2008 crisis, and much more so for bonds with speculative grade.

In our model, the endogenous inter-dependence between the liquidity and default premia for corporate bonds captures this important empirical regularity. By endogenizing the secondary market liquidity, our model points out that the origin of shock to liquidity premia can be traced back to the deterioration of firm fundamental itself. Thus, both default premium and liquidity premium are inter-dependent, and the positive feedback loop further amplifies and reinforces both premia in a nontrivial way.

5 Extensions and Discussions

5.1 Optimal Debt Maturity

Beyond the feedback loop between fundamental and liquidity, the debt maturity features a natural trade-off between liquidity provision and earlier inefficient default. This natural trade-off allows us to derive the optimal debt maturity (given the stationary maturity structure). Segura and Suarez (2011) present a related trade-off in a banking model without secondary markets but with periodic disruptions of the primary market for debt funding. Although the probability of these disruptions is *exogenous*, the severity of the disruptions is determined by how short the bank's maturity structure is. This is traded off against short-term debt being cheaper outside crisis states. In contrast, our model features an *endogenous* probability of default that is driven by the maturity structure and we also trade this off against cheaper short-term debt away from the bankruptcy boundary.

5.1.1 Liquidity provision: the bright side of short maturity

Section 3.4 has shown that bonds with shorter maturity have a more liquid secondary market, suggesting the role of liquidity provision for short-term debt. The efficiency gain due to short-term maturity arises from two channels.

First, debt holders hit by liquidity shocks become inefficient holders of bonds, and due to trading frictions the inefficient holding lasts for a while. As detailed in Appendix A.5, the steady-state proportion of L types as the firm is able to issue to only H/S types is given by

$$\mu_L(T) = \frac{\xi}{\lambda+\xi} - \underbrace{\frac{\xi\left[1-e^{-T(\lambda+\xi)}\right]}{T(\lambda+\xi)^2}}_{\text{Allocative efficiency}}, \tag{16}$$

with $\mu_L'(T) > 0$, $\lim_{T\to\infty}\mu_L(T) = \frac{\xi}{\lambda+\xi}$ and $\lim_{T\to 0}\mu_L(T) = 0$. Hence, the second term in (16) is the *allocative efficiency gain* of shortening the bond maturity T. Intuitively, shortening maturity alleviates this inefficiency because of the firm's superior primary market liquidity: whenever debt matures, the firm moves debt from inefficient L investors to efficient H investors via new bond issuance.[35]

[35]The firm could, instead of providing liquidity via maturity, allow bondholders with liquidity shocks to put back their bonds at the face value p. There are two important drawbacks. First, if the firm cannot distinguish who was hit by a liquidity shock, whenever $D_H < p$ all H investors will put back their debt at the same time. In fact, the put provision is akin to making bonds demand deposits and we are at traditional models of bank runs. Second, even if the liquidity shock is observable, there will be an additional flow term $\xi[D_H - p]\,dt$ as L investors are putting back their bonds to the firm every instant. This additional refinancing losses may influence the bankruptcy boundary in an adverse way and destroy the liquidity thus provided. The full implications of expanded bond contract terms is

Second, a shorter maturity reduces the rent extracted by dealers in the secondary market, thus leading to a *bargaining efficiency gain*. Intuitively, a shorter maturity, by allowing L investors to receive principal payment earlier, raises their outside option of waiting and in turn lowers the dealer's rent.

5.1.2 Earlier default: the dark side of short maturity

On the other hand, as first shown in LT96 (and formally proven in He and Xiong (2012b) and Diamond and He, 2012 [36]), shorter debt maturity in an LT96 style model leads to earlier default and thus greater dead-weight bankruptcy cost. In other words, the optimal maturity in LT96 and He and Xiong (2012b) is $T^* = \infty$, so that debt should always take the form of an infinitely lived consol bond. As discussed, the equity holders' rollover losses are $\frac{1}{T}\left[(1-\kappa)\, D_H(\delta, T) - P\right]$. In bad times (low fundamental δ), notwithstanding the fact that short-term debt has a greater market price $D_H(\delta, T)$, the effect of a higher rollover frequency $1/T$ dominates, leading to heavier rollover losses. As a result, equity holders default earlier if the firm is using shorter maturity debt.

5.1.3 Optimal Interior Debt Maturity

Relative to LT96 model where the debt maturity affects the equity holders' default decision, in our model the firm—being short of intermediating the market for its debt itself—could possibly use the inefficient tool of the maturity structure to provide liquidity services to bondholders. The inherent trade-off between liquidity provision and bankruptcy risk stemming from the maturity structure can then lead to an interior optimal $T^* < \infty$.

As a change to our benchmark model, we set $c = r = 2\%$. This is to remove a possible mechanical relationship between leverage and maturity structure T, as when $c \neq r$ changes in T would change leverage even in the risk-free case.

In Figure 5 we draw the optimal maturity T^* as a function of quasi leverage QL. The solid line

left for future work.

[36] He and Xiong (2012b) prove this claim for given (c, p) in the LT96 framework, while Diamond and He, 2012 prove this claim controlling for leverage (adjusting (c, p) to maintain the same debt value as shifts in the bankruptcy boundary caused by maturity shortening move the value of debt) in the random maturity framework of Leland (1998).

[NEW GRAPH OPTIMAL MATURITY HERE]

Figure 5: Left panel: **Total firm value** in the main model (solid line) and without frictions in the LT 96 model (dashed line). Right panel: **Optimal maturity** T^* in the main model as a function of quasi leverage.

depicts the optimal maturity for a secondary market with baseline intermediation, i.e., $\lambda = 15$For low (high) initial leverage, bankruptcy becomes more (less) remote, and the effect of liquidity provision (bankruptcy cost) dominates, resulting in a shorter (longer) optimal debt maturity. For a poorly intermediated market the firm for any QL is more willing to provide liquidity services via a short maturity structure than in a better intermediated market. In other words, a better functioning secondary market reduces the need to provide liquidity via shorter maturity and thus alleviates the bankruptcy pressure generated by the short debt structure.

5.2 Discussion of Asymmetric Information

In our model, the important driving force behind the spiking illiquidity near default is that there is a significant valuation wedge between H and L type investors for defaulted bonds, as summarized by the individual recovery values α_H and α_L. In the literature as well as in practice, an equally compelling explanation for the deteriorating liquidity of corporate bonds near default is a possibly worsening adverse selection problem due to information asymmetry. More specifically, one can imagine that some bond investors have private information regarding the bond's recovery value in default. As the firm edges closer to default, the informed agent's information becomes more valuable and he is more likely to attempt to sell his bonds. Thus, to guard against such adversely selected investors, a market maker in the Glosten and Milgrom (1985) tradition would raise the bid-ask spread.

Modeling such persistent adverse selection with long-lived bond investors, however, requires a lot more technical apparatus and thus awaits future research. To the extent that an adverse-selection-based model could conceivably lead to a similar qualitative result if asymmetric information is concentrated in the bond's recovery value,[37] then on the quantitative front our model has the

[37]If, instead, the adverse selection might not necessarily worsen when the firm goes closer to the default boundary, we would not expect, absent the bargaining frictions presented in this article, a monotonically increasing pattern of

advantage of incorporating standard structural bond valuation models in a simpler setting but still delivering the first-order empirical patterns.

6 Conclusion

We investigate the liquidity-fundamental spiral in the corporate bond market, by studying the endogenous liquidity of defaultable bonds in a search-based OTC markets together with the endogenous default decision by equity holders from the firm side.

By solving a system of PDEs, we derive the endogenous secondary market liquidity jointly with the debt valuations, equity valuations, and endogenous default policy, in closed-form. The fundamentals of corporate bonds, which are mainly driven by the firm's distance-to-default, affect the endogenous liquidity of corporate bonds.

The rollover channel that exposes equity holders to repricing of bonds, the fact that liquidity of corporate bonds worsens at the same time as the cash-flow fundamentally significantly hurts the equity holders' option value of keeping the firm alive. As a result, illiquidity of secondary corporate bond market feeds back to the fundamental of corporate bonds by edging the firm closer to bankruptcy. We hope our fully solved structural model can pave the way of bringing more structural approach in the empirical study of the impact of liquidity on corporate bonds.

In earlier working paper versions, we further incorporate endogenous firm investment and show that this mechanism, i.e., a feedback loop between the firm fundamental and the firm's (debt) financing liquidity, should encompass a broader set of firm level decisions beyond default.

illiquidity towards default.

References

AFONSO, G., AND R. LAGOS (2011): "Trade Dynamics in the Market for Federal Funds," *Working Paper*.

AMIHUD, Y., AND H. MENDELSON (1986): "Asset pricing and the bid-ask spread," *Journal of Financial Economics*, 17, 223–249.

BACK, K., AND S. BARUCH (2004): "Information in securities markets: Kyle meets Glosten and Milgrom," *Econometrica*, 72(2), 433–465.

BAO, J., J. PAN, AND J. WANG (2011): "The Illiquidity of Corporate Bonds," *Journal of Finance*, 66(3), 911–946.

BEBER, A., M. BRANDT, AND K. KAVAJECZ (2009): "Flight-to-quality or flight-to-liquidity? Evidence from the Euro-area bond market," *Review of Financial Studies*, 22, 925–957.

BHAMRA, H. S., L.-A. KUEHN, AND I. A. STREBULAEV (2010): "The Levered Equity Risk Premium and Credit Spreads: A Unified Framework," *Review of Financial Studies*, 23(2), 645–703.

BIAIS, B., AND P.-O. WEILL (2009): "Liquidity shocks and order book dynamics," *Working Paper*.

BLACK, F., AND J. C. COX (1976): "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance*, 31(2), 351–367.

BRUNNERMEIER, M. K., AND L. H. PEDERSEN (2009): "Market liquidity and funding liquidity," *Review of Financial Studies*, 22(6), 2201–2238.

CHEN, H. (2011): "Macroeconomic conditions and the puzzles of credit spreads and capital structures," *Journal of Finance*.

CHEN, L., D. LESMOND, AND J. WEI (2007): "Corporate yield spreads and bond liquidity," *Journal of Finance*, 62, 119–149.

Cheng, I.-H., and K. Milbradt (2012): "The hazards of debt: Rollover Freezes, Incentives, and Bailouts," *Review of Financial Studies*, 25(4), 1070–1110.

Choi, J., D. Hackbarth, and J. Zechner (2012): "Granularity of Corporate Debt : Theory and Tests," .

Cui, R., H. Chen, Z. He, and K. Milbradt (2013): "Liquidity and default of corporate bonds over the business cycle," *Working Paper*.

Custodio, C., M. Ferreira, and L. Laureano (????): "Why are US firms using more short term debt?," *Journal of Financial Economics*.

Dastidar, S., and B. Phelps (2009): "Introducing LCS: Liquidity Cost Scores for US Credit Bonds," *Barclay Capital Report*.

Diamond, D. W. (1993): "Seniority and maturity of debt contracts," *Journal of Financial Economics*, 33, 341–368.

Diamond, D. W., and Z. He (2012): "A theory of debt maturity: the long and short of debt overhang," *Working Paper*.

Dick-Nielsen, J., P. Feldhutter, and D. Lando (2012): "Corporate bond liquidity before and after the onset of the subprime crisis," *Journal of Financial Economics*, 103, 471–492.

Duffie, D., N. Garlenau, and L. H. Pedersen (2005): "Over-the-Counter Markets," *Econometrica*, 73(6), 1815–1847.

——— (2007): "Valuation in Over-the-Counter Markets," *Review of Financial Studies*, 20(6), 1865–1900.

Duffie, D., and D. Lando (2001): "Term structures of credit spreads with incomplete accounting information," *Econometrica*, 69(3), 633–664.

Edwards, A., L. Harris, and M. Piwowar (2007): “Corporate Bond Market Transaction Costs and Transparency,” *Journal of Finance*, 62, 1421–1451.

Ericsson, J., and O. Renault (2006): “Liquidity and credit risk,” *Journal of Finance*, p. 2219.

Feldhutter, P. (2011): “The same bond at different prices : identifying search frictions and selling pressures,” *Review of Financial Studies*.

Friewald, N., R. Jankowitsch, and M. G. Subrahmanyam (2012): “Illiquidity or credit detrioration: A study of liquidity in the US corporate bond market during financial crises,” *Journal of Financial Economics*, pp. 18–36.

Glosten, L. R., and P. R. Milgrom (1985): “Bid, ask and transaction prices in a specialist market with heterogeneously informed traders,” *Journal of Financial Economics*, 14(71-100).

Goldstein, I., E. Ozdenoren, and K. Yuan (2011): “Trading Frenzies and Their Impact on Real Investment,” *Working Paper*.

Green, R. C., B. Hollifield, and N. Schurhoff (2007a): “Dealer intermediation and price behavior in the aftermarket for new bond issues,” *Journal of Financial Economics*, 86, 643–682.

——— (2007b): “Financial intermediation and the costs of trading in an opaque market,” *Review of Financial Studies*, 20(2), 275–314.

Guerrieri, V., R. Shimer, and R. Wright (2010): “Adverse Selection in Competitive Search Equilibrium,” *Econometrica*, 78(6), 1823–1862.

Harris, L., and M. Piwowar (2006): “Secondary Trading Costs in the Municipal Bond Market,” *Journal of Finance*, 61, 1361–1397.

He, Z., and W. Xiong (2012a): “Dynamic debt runs,” *Review of Financial Studies*, (Forthcoming).

——— (2012b): “Rollover Risk and Credit Risk,” *Journal of Finance*, 67, 391–429.

Hong, G., and A. Warga (2000): "An Empirical Study of Bond Market Transactions," *Financial Analyst Journal*, 56, 32–46.

Huang, J.-Z., and M. Huang (2003): "How much of the corporate-treasury spread is due to credit risk?," *Working Paper*.

Kyle, A. S. (1985): "Continuous auctions and insider trading," *Econometrica*, 53(6), 1315–1335.

Lagos, R., and G. Rocheteau (2007): "Search in asset markets: Market structure, liquidity, and welfare," *American Economic Review*, 97(2), 198–202.

Lagos, R., and G. Rocheteau (2009): "Liquidity in asset markets with search frictions," *Econometrica*, 77(2), 403–426.

Lauermann, S., and A. Wolinsky (2011): "Search with Adverse Selection," *Working Paper*.

Leland, H. (1994): "Corporate debt value, bond covenants, and optimal capital structure," *Journal of Finance*, 49(4), 1213–1252.

——— (1998): "Agency costs, risk management, and capital structure," *Journal of Finance*, 53(4), 1213–1243.

Leland, H., and K. B. Toft (1996): "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads," *Journal of Finance*, 51(3), 987–1019.

Longstaff, F. A., S. Mithal, and E. Neis (2005): "Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Swap Market," *Journal of Finance*, 60, 2213–2253.

Manso, G. (2011): "Feedback Effects of Credit Ratings," *Working Paper*.

Miller, M. H. (1977): "Debt and taxes," *Journal of Finance*, 32, 261–275.

Morris, S., and H. S. Shin (2009): "Illiquidity component of credit risk," *Working Paper*.

Schultz, P. (2001): "Corporate Bond Trading Costs: A Peek Behind the Curtain," *Journal of Finance*, 56, 677–698.

Schwarz, K. (2010): "Mind the Gap: Disentangling Credit and Liquidity in Risk Spreads," *Working Paper*.

Segura, A., and J. Suarez (2011): "Dynamic maturity transformation," *Working Paper*.

Vayanos, D., and P.-O. Weill (2008): "A Search-Based Theory of the On-the-Run Phenomenon," *Journal of Finance*, 53(3), 1361–1398.

Weill, P.-O. (2007): "Leaning against the wind," *Review of Economic Studies*, 74, 1329–1354.

A Appendix

A.1 Notation

First, let us call $r_H \equiv r$, $r_L \equiv \overline{r}$ (where $\overline{r}$ is possibly different from r), $\xi_H \equiv \xi$ and $\xi_L \equiv \lambda\beta$, and $\tilde{\mu} = \mu - \frac{\sigma^2}{2}$. Second, define the log-transform $y = \log(\delta)$ so that $dy = \tilde{\mu}dt + \sigma dZ$. Third, for brevity we use the notation $D' \equiv \frac{\partial D}{\partial \delta}$ and $\dot{D} \equiv \frac{\partial D}{\partial \tau}$. We will, with abuse of notation, write $q(y, ...)$ to mean $\frac{y_b - y + ...}{...}$. Let $N(x)$ be the cumulative normal function. We will use $d_H(y, T)$ as the debt value in terms of the log-cash flow, so that $d_H(y, T) = D_H(e^y, T)$. We allow for general recovery value $\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 \frac{\delta_b}{r - \mu}$. In the text, we have $\boldsymbol{\alpha}_0 = \mathbf{0}$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1$.

A.2 2x2 matrix formulas

As the 2x2 specification is frequently used in the text, we present the results here in compact form. Suppose

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $\mathbf{A} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}$ where

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \mathbf{P} &= \begin{bmatrix} 1 & \frac{b}{\hat{r}_2 - a} \\ \frac{c}{\hat{r}_1 - d} & 1 \end{bmatrix} \\ \hat{\mathbf{D}} &= \begin{bmatrix} \hat{r}_1 & 0 \\ 0 & \hat{r}_2 \end{bmatrix}, \end{aligned}$$

where of course alternative versions of $\mathbf{P}$ can be chosen. However, to show convergence to frictionless markets we chose this form of $\mathbf{P}$ as it allows convergence to an upper triangular form. The roots

$$\begin{aligned} \hat{r}_{1/2} &= \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2} \\ &= \frac{a + d \pm \sqrt{(a - d)^2 + 4bc}}{2} \end{aligned}$$

solve $\det[\mathbf{A} - \rho\mathbf{I}] = 0$, i.e. $\hat{r}_{1/2}$ are the roots of the characteristic polynomial

$$g(\hat{r}) = (a - \hat{r})(d - \hat{r}) - bc = \hat{r}^2 - (a + d)\hat{r} + (ad - bc).$$

If $a > 0$ and $d > 0$ and $b < 0$ and $c < 0$ as well as $(ad - bc) > 0$, then both roots $\hat{r}_{1/2} > 0$.

Identifying $a = r_H + \xi_H, b = -\xi_H, c = -\xi_L, d = r_L + \xi_L$, we have

$$\hat{r}_i = \frac{r_H + r_L + \xi_H + \xi_L - (-1)^i \sqrt{[(r_H + \xi_H) - (r_L + \xi_L)]^2 + 4\xi_H\xi_L}}{2}.$$

We can also derive bounds on $\hat{r}_i$ by noting the following results:

$$\begin{aligned} g(r_H) &= \xi_H(r_L - r_H) > 0 \\ g(r_L) &= -\xi_L(r_L - r_H) < 0 \\ g(r_H + \xi_H) &= -\xi_H\xi_L < 0 \\ g(r_L + \xi_L) &= -\xi_H\xi_L < 0 \\ g(r_H + \xi_H + \xi_L) &= -\xi_L(r_L - r_H) < 0 \\ g(r_L + \xi_H + \xi_L) &= \xi_H(r_L - r_H) > 0 \end{aligned}$$

so that we know that

$$\begin{aligned} r_H &< \hat{r}_1 < \min\{r + \xi_H, r_L\} \\ \max\{r_H + \xi_H + \xi_L, r_L + \xi_L\} &< \hat{r}_2 < r_L + \xi_H + \xi_L. \end{aligned}$$

It is easy to show that as $\xi_H \to 0$, $\hat{r}_1 = r_L + \xi_L$ and $\hat{r}_2 = r_H$, and $\lim_{b \to 0} \mathbf{P} = \begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix}$, so that D_H converges towards the LT96 solution.

Next, consider $\lambda \to \infty$ such that $\xi_L \to \infty$, that is, what happens when the market becomes very liquid. Note that we can rewrite the characteristic polynomial as

$$g(\hat{r}) = \xi_L \left[(r_H + \xi_H - \hat{r}) \left(\frac{r_L}{\xi_L} + 1 - \frac{\hat{r}}{\xi_L} \right) - \xi_H \right]$$

Suppose now that $\hat{r}$ is finite. Then we know that the square bracket, as $\xi_L \to \infty$, becomes

$$(r_H + \xi_H - \hat{r}) - \xi_H = 0$$

so that $\hat{r}_2 = r_H > 0$. Thus, as both roots are positive, we must have that the second root $\hat{r}_1 \to \infty$. The diagonal decomposition becomes unstable, in that $\lim_{\lambda \to \infty} \mathbf{P} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Finally, for $r = r_H = r_L$ we can show that $\mathbf{P}^{-1}\mathbf{1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that $\hat{\mathbf{c}} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, and for $\alpha = \alpha_H = \alpha_L$ we have $\hat{\boldsymbol{\alpha}} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$.

A.3 Proofs of Section 3

A.3.1 Debt

Proof of Proposition 1.

Applying the log transform $y = \log(\delta)$ to the system of PDEs we are left with a linear system of PDEs:

$$\begin{aligned} \begin{bmatrix} r_H + \xi_H & -\xi_H \\ -\xi_L & r_L + \xi_L \end{bmatrix} \begin{bmatrix} d_H \\ d_L \end{bmatrix} &= \begin{bmatrix} c \\ \rho c \end{bmatrix} + \tilde{\mu} \begin{bmatrix} d_H \\ d_L \end{bmatrix}' + \frac{\sigma^2}{2} \begin{bmatrix} d_H \\ d_L \end{bmatrix}'' - \begin{bmatrix} \dot{d}_H \\ d_L \end{bmatrix} \\ \iff \mathbf{A} \times \mathbf{d} &= \mathbf{c} + \tilde{\mu}\mathbf{d}' + \frac{\sigma^2}{2}\mathbf{d}'' - \dot{\mathbf{d}} \end{aligned}$$

Here we allow for general changes to the coupon payment c by premultiplying by a parameter $\rho \leq 1$ to acknowledge that there might be linear holding costs above and beyond the higher discount rate. In the paper, we have $\rho = 1$. Let us decompose $\mathbf{A} = \mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}$ where $\hat{\mathbf{D}}$ is a diagonal matrix with its diagonal elements the eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is a matrix of the respective stacked eigenvectors. The resulting eigenvalues are defined

$$g(\hat{r}) = (r_H + \xi_H - \hat{r})(r_L + \xi_L - \hat{r}) - \xi_L \xi_H = 0$$

and $g(r_H) = \xi_H (r_L - r_H) > 0$ and $g(r_L) = -\xi_L (r_L - r_H) < 0$. We thus have $\hat{r}_i = \frac{r+\xi+\overline{r}+\lambda\beta \pm \sqrt{[(r+\xi)-(\overline{r}+\lambda\beta)]^2 + 4\xi\lambda\beta}}{2}$.

Premultiplying the system by $\mathbf{P}^{-1}$ and noting that $\mathbf{P}^{-1}\mathbf{A} = \hat{\mathbf{D}}\mathbf{P}^{-1}$ we have a delinked system PDEs with a common bankruptcy boundary $y_b \equiv \log(\delta_b)$ and payout boundary $t = 0$

$$\begin{aligned} \hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{d} &= \mathbf{P}^{-1}\mathbf{c} + \tilde{\mu}\mathbf{P}^{-1}\mathbf{d}' + \frac{\sigma^2}{2}\mathbf{P}^{-1}\mathbf{d}'' - \mathbf{P}^{-1}\dot{\mathbf{d}} \\ \iff \hat{\mathbf{D}}\hat{\mathbf{d}} &= \widehat{\mathbf{c}} + \tilde{\mu}\hat{\mathbf{d}}' + \frac{\sigma^2}{2}\hat{\mathbf{d}}'' - \dot{\hat{\mathbf{d}}} \end{aligned}$$

where $\hat{\mathbf{d}} = \mathbf{P}^{-1}\mathbf{d}$ and $\hat{\mathbf{c}} = \mathbf{P}^{-1}\mathbf{c}$. The rows of the system are now delinked, and we are left with two PDEs of the form

$$\hat{r}_i \hat{d}_i = \widehat{c}_i + \tilde{\mu}\hat{d}_i' + \frac{\sigma^2}{2}\hat{d}_i'' - \dot{\hat{d}}_i$$

with given boundary conditions at $t = 0$ and $y = y_b$, whose solutions are known from LT96. The decomposition works because the boundaries are the same across rows. The solution takes the form

$$\begin{aligned}
\hat{d}_i &= \left(\hat{\mathbf{k}}_0^D\right)_i + \left(\hat{\mathbf{k}}_F^D\right)_i e^{-\hat{r}_i t}\left(1 - F_i\right) + \left(\hat{\mathbf{k}}_G^D\right)_i G_i \\
F_j\left(y, t\right) &= \sum_{i=1}^{2} e^{\left(y - y_b\right)\varphi_{ij}} N\left[q\left(y, \varphi_{ij}, t\right)\right] \\
G_j\left(y, t\right) &= \sum_{i=1}^{2} e^{\left(y - y_b\right)\gamma_{ji}} N\left[q\left(y, \gamma_{ji}, t\right)\right]
\end{aligned}$$

where

$$q\left(y, \rho, t\right) = \frac{y_b - y - \left(\rho + a\right)\cdot\sigma^2 t}{\sigma\sqrt{t}}$$

and constants

$$\begin{aligned}
\left(\hat{\mathbf{k}}_0^D\right)_i &= \frac{\hat{c}_i}{\hat{r}_i} \\
\left(\hat{\mathbf{k}}_F^D\right)_i &= \left(\hat{p}_i - \frac{\hat{c}_i}{\hat{r}_i}\right) \\
\left(\hat{\mathbf{k}}_G^D\right)_i &= \left(\hat{\alpha}_i \frac{e^{y_b}}{r - \mu} - \frac{\hat{c}_i}{\hat{r}_i}\right)
\end{aligned}$$

and some yet to be determined parameters $\varphi_{ij}, \gamma_{ji}$. Note that $\lim_{t\to 0} q\left(y, \rho, t\right) = \lim_{t\to 0} \frac{y_b - y}{\sigma\sqrt{t}} = -\infty$ as $y_b < y$, so $N\left[q\left(y, \rho, 0\right)\right] = 0$ for all i and $y > y_b$. Further note that $\lim_{y\to\infty} q\left(y, \rho, t\right) = -\infty$, so $\lim_{y\to\infty} N\left[q\left(y, \rho, t\right)\right] = 0$. Substituting the candidate solution $\hat{d}_i$ into the PDE with $\left(\hat{\mathbf{k}}_0^D\right)_i = \widehat{\frac{c_i}{\hat{r}_i}}$, $\left(\hat{\mathbf{k}}_F^D\right)_i = \hat{p}_i - \frac{\hat{c}_i}{\hat{r}_i}$, $\left(\hat{\mathbf{k}}_G^D\right)_i = \hat{\alpha}_i \frac{\exp(y_b)}{r-\mu} - \frac{\hat{c}_i}{\hat{r}_i}$, we see that

$$\begin{aligned}
b_i e^{-\hat{r}_i t}\left[\hat{r}_i\left(1 - F_i\right) + \tilde{\mu}F_i' + \frac{\sigma^2}{2}F'' - \left[\hat{r}_i\left(1 - F_i\right) + \dot{F}_i\right]\right] & \\
+ c_i\left[\hat{r}_i G_i - \tilde{\mu}G_i' - \frac{\sigma^2}{2}G_i'' + \dot{G}_i\right] &= 0 \\
\iff b_i e^{-\hat{r}_i t}\left[\tilde{\mu}F' + \frac{\sigma^2}{2}F'' - \dot{F}\right] & \\
+ c_i\left[\hat{r}_i G_i - \tilde{\mu}G_i' - \frac{\sigma^2}{2}G_i'' + \dot{G}_i\right] &= 0
\end{aligned}$$

We see that both $\dot{F}_i$ and $\dot{G}_i$ have no term $N\left(\cdot\right)$. As q is linear in y, we have $q'' = 0$ (where $q' = q_y$ and $\dot{q} = q_t$). We thus have, for F,

$$\begin{aligned}
N\left[q\left(y, \varphi, t\right)\right]\left[\tilde{\mu}\varphi + \frac{\sigma^2}{2}\varphi^2\right] & \\
+ \phi\left[q\left(v, \varphi, t\right)\right]\left[\tilde{\mu}q' + \frac{\sigma^2}{2}\left[2\varphi q' - q\left(q'\right)^2\right] - \dot{q}\right] &= 0
\end{aligned}$$

So the roots for F_i are $\varphi_1 = 0 = -a + a$ and $\varphi_2 = -\frac{2\tilde{\mu}}{\sigma^2} = -a - a$ where $a \equiv \frac{\tilde{\mu}}{\sigma^2}$. We see that this is independent of i, that is, it is independent of what row of $\hat{\mathbf{d}}$ we picked, as $\hat{r}_i$ is cancelled out. Further, for G, we have

$$\begin{aligned}
N\left[q\left(v, \gamma, t\right)\right]\left[\tilde{\mu}\gamma + \frac{\sigma^2}{2}\gamma^2 - \hat{r}_i\right] & \\
+ \phi\left[q\left(v, \gamma, t\right)\right]\left[\tilde{\mu}q' + \frac{\sigma^2}{2}\left[2\gamma q' - q\left(q'\right)^2\right] - \dot{q}\right] &= 0
\end{aligned}$$

so the roots for G_i are $\gamma_{i1} = \frac{-\tilde{\mu} + \sqrt{\tilde{\mu}^2 + 2\sigma^2\hat{r}_i}}{\sigma^2} = -a + \frac{\sqrt{\tilde{\mu}^2 + 2\sigma^2\hat{r}_i}}{\sigma^2}$ and $\gamma_{i2} = -a - \frac{\sqrt{\tilde{\mu}^2 + 2\sigma^2\hat{r}_i}}{\sigma^2}$. Simply plugging in the functional form of q results in the term in square brackets in the second row to vanish.

For the boundary condition, we have

$$\begin{aligned}\hat{\mathbf{d}}(y,0) &= \mathbf{P}^{-1}\mathbf{1}\cdot p = \hat{\mathbf{p}}\\ \hat{\mathbf{d}}(y_b,t) &= \mathbf{P}^{-1}\boldsymbol{\alpha}\frac{\exp(y_b)}{r-\mu} = \hat{\boldsymbol{\alpha}}\frac{\exp(y_b)}{r-\mu}\end{aligned}$$

which defines the remaining parameters of the solution.

As a last step, we retranslate the system back into the original debt functions by premultiplying by $\mathbf{P}$ and noting that $F(v,t) = F_i(v,t) = F_{-i}(v,t)$ by the symmetry of the φ's, and by rewriting it in terms of $\delta = \exp(y)$. ■

A.3.2 Equity

Proof of Proposition 2.

Equity has the following ODE where (for notational ease we define $m = 1/T$)

$$rE = \exp(y) - (1-\pi)c + \tilde{\mu}E' + \frac{\sigma^2}{2}E'' + m\left[D_H(y,T) - p\right]$$

The term in square brackets is the cash-flow term that arises out of rollover of debt (while keeping coupon, principal and maturity stationary), a term first pointed out by LT96. We will establish the (closed-form) solution in several steps.

First, the homogenous solutions to the ODE are $M(y) = e^{\eta_1 y}$ and $U(y) = e^{\eta_2 y}$ where

$$\frac{\sigma^2}{2}\eta^2 + \tilde{\mu}\eta - r = 0$$

so that

$$\eta_{1/2} = \frac{-\tilde{\mu} \pm \sqrt{\tilde{\mu}^2 + 2\sigma^2 r}}{\sigma^2} = -a \pm \frac{\sqrt{\tilde{\mu}^2 + 2\sigma^2 r}}{\sigma^2}$$

and $\eta_1 > 1 > 0 > \eta_2$.

Next, let us establish the Wronskian

$$\begin{aligned}Wr(s) &= M(s)U'(s) - M'(s)U(s)\\ &= -(\eta_1 - \eta_2)\exp\{(\eta_1+\eta_2)s\}\\ &= -\Delta\eta \cdot M(s)U(s)\end{aligned}$$

Then, by the variation of coefficient solutions to linear ODEs, a technique described in most textbooks on differential equations, we have for an ODE

$$rg = \tilde{\mu}g' + \frac{\sigma^2}{2}g'' + part(s)$$

the following particular solution g_p

$$\begin{aligned}g_p(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{M(s)U(x) - M(x)U(s)}{Wr(s)}ds\\ &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{e^{-\eta_2 s}e^{\eta_2 x} - e^{\eta_1 x}e^{-\eta_1 s}}{-\Delta\eta}ds\\ g_p'(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{M(s)U'(x) - M'(x)U(s)}{Wr(s)}ds\\ &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{\eta_2 M(s)U(x) - \eta_1 M(x)U(s)}{Wr(s)}ds\\ g_p''(x|l) &= \frac{2}{\sigma^2}\int_x^l part(s)\frac{\eta_2^2 M(s)U(x) - \eta_1^2 M(x)U(s)}{Wr(s)}ds - \frac{2}{\sigma^2}part(x)\end{aligned}$$

for an arbitrary limit $l \in (v_B, \infty)$.

Second, as the debt term D_H is bounded, to impose the condition that equity does not grow orders of magnitude faster than the unlevered value of the firm $V(y) = \frac{e^y}{r-\mu}$ we need $\lim_{y\to\infty}\left|\frac{E(y)}{V(y)}\right| < \infty$. Let us write the solution as

$$E(y) = k_U^E U(y) + k_M^E M(y) + V(y) + k_0^E + \int_y^l \frac{2}{\sigma^2}part(s)\frac{M(s)U(y) - M(y)U(s)}{Wr(s)}ds$$

where we incorporated all constant terms of the ODE into the definition of k_0^E and $part(s)$ is thus just composed of cumulative normal functions of the form $N[-aa\cdot y+bb]$ where $aa>0$. Let us gather terms of $U(y)$ and $M(y)$ to get

$$E(y)=U(y)\left[k_U^E+\int_y^l \frac{2}{\sigma^2}part(s)\frac{M(s)}{Wr(s)}ds\right]+M(y)\left[k_M^E-\int_y^l \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds\right]+\frac{e^y}{r-\tilde{\mu}}+k_0^E$$

First, let us note that the integrals all converge, as $N[-aa\cdot y+bb]$ converges faster than any function $e^{cst\cdot y}$ for any constant cst. Second, to impose the boundary condition of $\lim_{y\to\infty}\left|\frac{E(y)}{V(y)}\right|<\infty$, we note that $\lim_{y\to\infty}U(y)=0$ so the first term in the above equation converges for any choice of K_U. However, the second term contains $M(y)$ which explodes to infinity faster than e^y as $\eta_1>1$. We thus need to pick

$$k_M^E(l)=-\int_l^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds$$

as a necessary condition to have the term stay bounded. Next, plugging it in, we see that the term in question becomes

$$M(y)\left[K_M(l)-\int_y^l \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds\right]=-M(y)\int_y^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds$$

and we now show that this term converges to 0 as $y\to\infty$. Let us rewrite to get

$$\begin{aligned}\lim_{y\to\infty}-M(y)\int_y^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds &= \lim_{y\to\infty}\frac{-\int_y^\infty \frac{2}{\sigma^2}part(s)\frac{U(s)}{Wr(s)}ds}{\frac{1}{M(y)}}=\frac{"0"}{"0"}\\ &\overset{\{L'Hopital\}}{=} \lim_{y\to\infty}\frac{\frac{2}{\sigma^2}part(y)\frac{U(y)}{Wr(y)}}{\frac{M'(y)}{[M(y)]^2}}=0\end{aligned}$$

and again, we see that since $U(y), Wr(y), M(y), M'(y)$ are all of exponential form and $part(y)$ is of cumulative normal form this term converges to zero rapidly, and the solution to $E(y)$ is verified. Let us take the arbitrary limit $l\to\infty$ and define $g_p(x)\equiv g_p(x|\infty)$. We note that the complement of the integrals (i.e. $\int_l^\infty \cdot ds$) vanishes, so that $\lim_{l\to\infty}K_M(l)=0$. We see that $g_p(x)$ and $g_p'(x)$ (and so forth) consists of a finite sum of integrals of the form $\int_x^\infty e^{cst\cdot s}N[q(s,\rho,T)]\,ds$ where cst is a constant.

Third, let us briefly establish two auxiliary results. First, let us note that for $aa>0$ we have

$$aa\int_x^\infty \phi(-aa\cdot s+bb)\,ds=\int_{-\infty}^{-aa\cdot x+bb}\phi(y)\,dy=N[-aa\cdot x+bb]$$

by simple change of variables. Second, note that

$$\begin{aligned}e^{cst\cdot x}\phi(-aa\cdot x+bb) &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[(-aa\cdot x+bb)^2-2cst\cdot x\right]\right\}\\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\left(-aa\cdot x+bb+\frac{cst}{aa}\right)^2+bb^2-\left(bb+\frac{cst}{aa}\right)^2\right]\right\}\\ &= \phi\left(-aa\cdot x+bb+\frac{cst}{aa}\right)e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)}\end{aligned}$$

by a simple completion of the square. Now, we can solve the integral in question via integration by parts:

$$\begin{aligned}&\int_x^\infty e^{cst\cdot s}N[-aa\cdot s+bb]\,ds\\ =\;& \frac{e^{cst\cdot s}}{cst}N[-aa\cdot s+bb]\Big|_{s=x}^{\infty}+\frac{1}{cst}\left[aa\cdot\int_x^\infty e^{cst\cdot s}\phi(-aa\cdot s+bb)\,ds\right]\\ =\;& -\frac{e^{cst\cdot x}}{cst}N[-aa\cdot x+bb]+\frac{1}{cst}\left[a\int_x^\infty \phi\left(-aa\cdot s+bb+\frac{cst}{aa}\right)ds\right]e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)}\\ =\;& -\frac{e^{cst\cdot x}}{cst}N[-aa\cdot x+bb]+\frac{1}{cst}N\left[-aa\cdot x+bb+\frac{cst}{aa}\right]e^{\frac{cst}{aa}\left(bb+\frac{1}{2}\frac{cst}{aa}\right)}\end{aligned}$$

where we again used the fact that the cumulative normal vanishes faster than any exponential function explodes. We also need

$$\begin{aligned}
& \int_x^\infty N\left[-aa\cdot s+bb\right]ds \\
= \quad & sN\left[-aa\cdot s+bb\right]\Big|_{s=x}^{\infty}+aa\int_x^\infty \phi\left(-aa\cdot s+bb\right)ds \\
= \quad & -xN\left[-aa\cdot x+bb\right]+\frac{1}{aa}\left\{\phi\left(-aa\cdot x+bb\right)+bb\cdot N\left[-aa\cdot x+bb\right]\right\} \\
= \quad & \frac{1}{aa}\left[\left(-aa\cdot x+bb\right)N\left[-aa\cdot x+bb\right]+\phi\left(-aa\cdot x+bb\right)\right]
\end{aligned}$$

which is essentially $\lim_{cst\to 0}\int_x^\infty e^{cst\cdot s}N\left[-aa\cdot s+bb\right]ds$.

Next, note that $D_i\left(y,t\right)=...+...e^{(y-y_b)\rho}N\left[q\left(y,\rho,t\right)\right]+...$ for some ρ, so that we are essentially facing integrals

$$\begin{aligned}
& \frac{2}{\sigma}\int_x^\infty e^{(s-y_b)\rho}N\left[q\left(s,\rho,t\right)\right]\frac{M\left(s\right)U\left(x\right)}{Wr\left(s\right)}ds \\
= \quad & \frac{2}{\sigma}\frac{1}{-\Delta\eta}e^{\eta_2 x}e^{-y_b\rho}\int_x^\infty e^{(\rho-\eta_2)s}N\left[q\left(s,\rho,t\right)\right]ds \\
= \quad & \frac{2}{\sigma}\frac{1}{-\Delta\eta}e^{\eta_2 x}e^{-y_b\rho}\frac{1}{\rho-\eta_2} \\
& \times\left[-e^{(\rho-\eta_2)}N\left[q\left(x,\rho,t\right)\right]+N\left[q\left(x,\eta_2,t\right)\right]e^{(\rho-\eta_2)\left\{y_b-\frac{1}{2}\left[(\eta+a)^2-(\rho+a)^2\right]\sigma^2 T\right\}}\right]
\end{aligned}$$

Here, we used $cst=\left(\rho-\eta_2\right)$, $aa=\frac{1}{\sigma\sqrt{T}}$, $b=\frac{y_b-(\rho+a)\sigma^2 T}{\sigma\sqrt{T}}$, $q\left(x,\rho,t\right)+\left(\rho-\eta\right)\sigma\sqrt{t}=q\left(x,\eta,t\right)$ and the fact that

$$\begin{aligned}
\left(\rho-\eta\right)\left(-\right)\left[\rho+a-\frac{1}{2}\left(\rho-\eta\right)\right] \quad &= \quad \left(\rho-\eta\right)\left(-\right)\left[\frac{1}{2}\rho+\frac{1}{2}a+\frac{1}{2}\eta+\frac{1}{2}a\right] \\
&= \quad \frac{1}{2}\left[\left(\eta+a\right)^2-\left(\rho+a\right)^2\right]
\end{aligned}$$

where we note that the last term is independent of if we pick the larger or smaller root, as both η and all possible ρ are centered around $-a$. Lastly, we note that $\frac{2}{\sigma}\int_x^\infty e^{(s-y_b)\rho}N\left[q\left(s,\rho,t\right)\right]\frac{M(x)U(s)}{Wr(s)}ds$ has the same form of solution only with η_1 replacing η_2. Define

$$\begin{aligned}
H\left(x,\rho,\eta,T\right) \quad &\equiv \quad \int_x^\infty e^{(\rho-\eta)\cdot s}N\left[q\left(s,\rho,T\right)\right]ds \\
&= \quad -\frac{1}{cst}\left\{e^{cst\cdot x}N\left[q\left(x,\rho,T\right)\right]-e^{cst\cdot y_b}\exp\left\{-cst\left(\rho+a-\frac{1}{2}cst\right)\sigma^2 T\right\}N\left[q\left(x,\rho,T\right)+cst\cdot\sigma\sqrt{T}\right]\right\} \\
&= \quad \frac{1}{\eta-\rho}\left\{e^{(\rho-\eta)x}N\left[q\left(x,\rho,T\right)\right]-e^{(\rho-\eta)y_b}e^{\frac{1}{2}\left[(\eta+a)^2-(\rho+a)^2\right]\sigma^2 T}N\left[q\left(x,\eta,T\right)\right]\right\}
\end{aligned}$$

if $\rho\neq\eta$, and define

$$\begin{aligned}
H\left(x,\rho,\eta,T\right) \quad &\equiv \quad \int_x^\infty e^{(\rho-\eta)\cdot s}N\left[q\left(s,\rho,T\right)\right]ds=\int_x^\infty N\left[q\left(s,\rho,T\right)\right]ds \\
&= \quad \sigma\sqrt{T}\left[q\left(s,\rho,T\right)N\left[q\left(s,\rho,T\right)\right]+\phi\left(q\left(s,\rho,T\right)\right)\right]
\end{aligned}$$

for $\rho=\eta$. Note that

$$H\left(y_b,\rho,\eta,T\right)=\begin{cases}\frac{e^{(\rho-\eta)y_b}}{\eta-\rho}\left\{N\left[-\left(\rho+a\right)\sigma\sqrt{T}\right]-e^{\frac{1}{2}\left[(\eta+a)^2-(\rho+a)^2\right]\sigma^2 T}N\left[-\left(\eta+a\right)\sigma\sqrt{T}\right]\right\} & ,\rho\neq\eta \\ \sigma\sqrt{T}\left[-\left(\rho+a\right)\sigma\sqrt{T}\cdot N\left[-\left(\rho+a\right)\sigma\sqrt{T}\right]+\phi\left(-\left(\rho+a\right)\sigma\sqrt{T}\right)\right] & ,\rho=\eta\end{cases}$$

The solution to the particular part for F then is

$$\begin{aligned}
g_F(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty F(s)\,\frac{M(s)\,U(x)-M(x)\,U(s)}{Wr(s)}ds \\
&= \frac{1}{-\Delta\eta}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{e^{\eta_2 x}e^{-\varphi_i y_b}H(x,\varphi_i,\eta_2,T)-e^{\eta_1 x}e^{-\varphi_i y_b}H(x,\varphi_i,\eta_1,T)\right\} \\
g_F'(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty F(s)\,\frac{\eta_2 M(s)\,U(x)-\eta_1 M(x)\,U(s)}{Wr(s)}ds \\
&= \frac{1}{-\Delta\eta}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{\eta_2 e^{\eta_2 x}e^{-\varphi_i y_b}H(x,\varphi_i,\eta_2,T)-\eta_1 e^{\eta_1 x}e^{-\varphi_i y_b}H(x,\varphi_i,\eta_1,T)\right\}
\end{aligned}$$

and the solution to the particular part for G_j is

$$\begin{aligned}
g_{G_j}(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty G_j(s)\,\frac{M(s)\,U(x)-M(x)\,U(s)}{Wr(s)}ds \\
&= \frac{1}{-\Delta\eta}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{e^{\eta_2 x}e^{-\gamma_{ji} y_b}H(x,\gamma_{ji},\eta_2,T)-e^{\eta_1 x}e^{-\gamma_{ji} y_b}H(x,\gamma_{ji},\eta_1,T)\right\} \\
g_{G_j}'(x) &\equiv \frac{2}{\sigma^2}\int_x^\infty G_j(s)\,\frac{\eta_2 M(s)\,U(x)-\eta_1 M(x)\,U(s)}{Wr(s)}ds \\
&= \frac{1}{-\Delta\eta}\frac{2}{\sigma^2}\sum_{i=1}^{2}\left\{\eta_2 e^{\eta_2 x}e^{-\gamma_{ji} y_b}H(x,\gamma_{ji},\eta_2,T)-\eta_1 e^{\eta_1 x}e^{-\gamma_{ji} y_b}H(x,\gamma_{ji},\eta_1,T)\right\}
\end{aligned}$$

Plugging in $x=y_b$, and noting that $q(y_b,\rho,t)=-(\rho+a)\,\sigma\sqrt{t}$, we make the important observation that

$$e^{\eta y_b}e^{-\rho y_b}H(y_b,\rho,\eta,T)=\frac{1}{\eta-\rho}\left\{N\left[-(\rho+a)\,\sigma\sqrt{T}\right]-e^{\frac{1}{2}\left[(\eta+a)^2-(\rho+a)^2\right]\sigma^2 T}N\left[-(\eta+a)\,\sigma\sqrt{T}\right]\right\}$$

is independent of y_b. We thus conclude that for any particular part $g_p(y_b)$, of the form given above, and its derivative $g_p'(y_b)$ are **independent** of y_b besides $C(y_b)$ containing e^{y_b}. Also note that for $\rho=\{\varphi_1,\varphi_2\}$ we have

$$e^{\frac{1}{2}\left[(\eta+a)^2-(\varphi+a)^2\right]\sigma^2 T}=e^{rT}$$

and for $\rho=\{\gamma_{i1},\gamma_{i2}\}$ we have

$$e^{\frac{1}{2}\left[(\eta+a)^2-\left(\gamma_{ji}+a\right)^2\right]\sigma^2 T}=e^{(r-\hat{r}_i)T}$$

Total equity is now easily written out to be

$$\begin{aligned}
E(y) &= k_2^E e^{\eta_2(y-y_b)}+\frac{e^y}{r-\mu}+k_0^E+g_p(y) \\
&= k_2^E e^{\eta_2(y-y_b)}+\frac{e^y}{r-\mu}+k_0^E+\frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[-\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F(y)+\mathbf{g}_G(y)\,\hat{\mathbf{k}}_G^D\right]
\end{aligned}$$

where we scaled the constants k_U^E by $e^{-\eta_2 y_b}$ so that $k_2^E=k_U^E\cdot e^{-\eta_2 y_b}$. The constant term k_0^E is

$$k_0^E=\frac{1}{r}\left\{-(1-\pi)\,c+\frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[\hat{\mathbf{k}}_0^D+\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D-p\right]\right\}$$

The constant K is derived by setting

$$\begin{aligned}
0=E(y_b) &= k_2^E+\frac{e^{y_b}}{r-\mu}+k_0^E+\frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[-\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F(y_b)+\mathbf{g}_G(y_b)\,\hat{\mathbf{k}}_G^D\right] \\
\iff k_2^E(y_b) &= -\left(\frac{e^{y_b}}{r-\mu}+k_0^E+\frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[-\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F(y_b)+\mathbf{g}_G(y_b)\,\hat{\mathbf{k}}_G^D\right]\right)
\end{aligned}$$

The term in brackets only features linear combinations of constants independent of y_b. ■

A.3.3 Optimal Default

Proof of Proposition 3.

The optimal $\delta_b = e^{y_b}$ is now easily derived. Plugging in $k_2^E(y_b)$ into the smooth pasting condition $E'(y_b) = 0$, we can derive $\delta_b = e^{y_b}$ in closed form:

$$\begin{aligned}
0 &= E'(y_b) \\
&= k_2^E(y_b)\,\eta_2 + \frac{e^{y_b}}{r-\mu} + \frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F'(y_b) + \mathbf{g}_G'(y_b)\,\hat{\mathbf{k}}_G^D\right] \\
&= \eta_2\left(\frac{e^{y_b}}{r-\mu} + k_0^E + \frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[-\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F(y_b) + \mathbf{g}_G(y_b)\,\hat{\mathbf{k}}_G^D\right]\right) \\
&\quad + \frac{e^{y_b}}{r-\mu} + \frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D g_F'(y_b) + \mathbf{g}_G'(y_b)\,\hat{\mathbf{k}}_G^D\right] \\
&= -\frac{e^{y_b}}{r-\mu}\left[\eta_2 - 1 + \frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left\{\eta_2\mathbf{g}_G(y_b) - \mathbf{g}_G'(y_b)\right\}\hat{\boldsymbol{\alpha}}_1\right] \\
&\quad -\eta_2 k_0^E + \frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left[\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D\left\{\eta_2 g_F(y_b) - g_F'(y_b)\right\} + \left\{\eta_2\mathbf{g}_G(y_b) - \mathbf{g}_G'(y_b)\right\}\left(\hat{\mathbf{k}}_0^D - \hat{\boldsymbol{\alpha}}_0\right)\right]
\end{aligned}$$

which yields

$$\begin{aligned}
\delta_b = e^{y_b} &= (r-\mu)\times\left[\eta_2 - 1 + \frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left\{\eta_2\mathbf{g}_G(y_b) - \mathbf{g}_G'(y_b)\right\}\hat{\boldsymbol{\alpha}}_1\right]^{-1} \\
&\quad \times\left[\begin{array}{c} -\eta_2 k_0^E + \frac{1}{T}\mathbf{S}\cdot\mathbf{P}\exp\left(-\hat{\mathbf{D}}T\right)\hat{\mathbf{k}}_F^D\left\{\eta_2 g_F(y_b) - g_F'(y_b)\right\} \\ +\frac{1}{T}\mathbf{S}\cdot\mathbf{P}\left\{\eta_2\mathbf{g}_G(y_b) - \mathbf{g}_G'(y_b)\right\}\left(\hat{\mathbf{k}}_0^D - \hat{\boldsymbol{\alpha}}_0\right)\end{array}\right]
\end{aligned}$$

where we note that the right hand side is independent of y_b by previous results. We can simplify further by noting that each of the terms in curly brackets can be written as

$$\begin{aligned}
&\eta_2 g_F(y_b) - g_F'(y_b) \\
= {}& \eta_2\frac{2}{\sigma^2}\int_{y_b}^{\infty} F(s)\,\frac{M(s)\,U(y_b) - M(y_b)\,U(s)}{Wr(y_b)}ds - \frac{2}{\sigma^2}\int_{y_b}^{\infty} F(s)\,\frac{\eta_2 M(s)\,U(y_b) - \eta_1 M(y_b)\,U(s)}{Wr(y_b)}ds \\
= {}& \frac{2}{\sigma^2}\int_{y_b}^{\infty} F(s)\,\frac{(\eta_1-\eta_2)\,M(y_b)\,U(s)}{Wr(y_b)}ds \\
= {}& -\frac{2}{\sigma^2}\sum_{i=1}^{2} e^{(\eta_1-\varphi_i)y_b} H(y_b,\varphi_i,\eta_1,T)
\end{aligned}$$

We thus established a closed form, albeit quite complex, for the optimal y_b.

The limit $\lim_{T\to\infty}\delta_b$ can be easily derived by noting that the normal distributions either converge to 0 or 1, so the only difficulty remaining is the term $e^{\frac{1}{2}\left[(\eta_1+a)^2-(\varphi_i+a)^2\right]\sigma^2 T}$. Let us establish a series of results:

First, we note that in addition to $e^{\frac{1}{2}\left[(\eta_1+a)^2-(\varphi_i+a)^2\right]\sigma^2 T} = e^{r_H T}$, we have

$$e^{\frac{1}{2}\left[(\eta_1+a)^2-\left(\gamma_{ji}+a\right)^2\right]\sigma^2 T} = e^{\left(r_H - \hat{r}_j\right)T}$$

and since we established that $\hat{r}_j > r_H$ we note that this term is converging to zero.

Second, we note that

$$\begin{aligned}
\lim_{T\to\infty}\frac{N\left[-(\eta_1+a)\,\sigma\sqrt{T}\right]}{e^{-r_H T}} &= \frac{"0"}{"0"} = \lim_{T\to\infty}\frac{\left(N\left[-(\eta_1+a)\,\sigma\sqrt{T}\right]\right)'}{\left(e^{-r_H T}\right)'} \\
&= \lim_{T\to\infty}\frac{(\eta_1+a)\,\sigma}{2r_H\sqrt{T}}\exp\left\{-\frac{1}{2}(\eta_1+a)^2\sigma^2 T + r_H T\right\} \\
&= \lim_{T\to\infty}\frac{(\eta_1+a)\,\sigma}{2r_H\sqrt{T}}\exp\left\{-T\left[\frac{\tilde{\mu}^2}{2\sigma^2} + r_H - r_H\right]\right\} \\
&= \lim_{T\to\infty}\frac{(\eta_1+a)\,\sigma}{2r_H\sqrt{T}}\exp\left\{-\frac{\tilde{\mu}^2}{2\sigma^2}T\right\} = 0
\end{aligned}$$

where we used the fact that $(\eta_1+a)^2 = \frac{\tilde{\mu}^2+2\sigma^2 r_H}{\sigma^4}$. Thus, all terms involving functions g vanish and no complication arises from premultiplying by $m = \frac{1}{T}$, and we are left with

$$\lim_{T\to\infty} \frac{\delta_b}{r-\mu} = \lim_{T\to\infty} V_B = \lim_{T\to\infty} \frac{-\eta_2 k_0^E(T)}{\eta_2 - 1} = \frac{\eta_2(1-\pi)c}{\eta_2 - 1}$$

where $V_B = \frac{\delta_b}{r-\mu}$ which is the same result as in Leland (1994) once we identify (in Leland's notation) $x = -\eta_2$, so that $\lim_{T\to\infty} V_B = \frac{(1-\pi)\frac{c}{r}x}{x+1}$. In the infinite maturity limit, the equity holders care about the illiquidity they impose on bondholders via the valuation spread between H and L only at the beginning when issuing bonds, but since there is no rollover their default decision is not affected by bond market illiquidity for a given level of aggregate face value and coupon.

Next, let us investigate $T \to 0$, which essentially renders the secondary bond market completely liquid. But of course there is a large effect of $T \to 0$ on the bankruptcy decision of the equity holders. Using L'Hopital's rule, we need to investigate

$$\lim_{T\to 0} \frac{1}{T}\left[\eta_2 g_F(v_B) - g_F'(v_B)\right]$$

We see that two terms that exactly give $\eta_i - \rho$ explode at the rate $\frac{1}{\sqrt{T}}$, so that in the limit we have

$$\lim_{T\to\infty} \frac{\delta_b(T)}{r-\mu} = \frac{\sum_{j=1}^2 P_{1j}(B_j + A_j)}{\sum_{j=1}^2 P_{1j}\hat{\alpha}_j} = \frac{p\begin{bmatrix} P_{11} & P_{12}\end{bmatrix}\mathbf{P^{-1}1}}{\begin{bmatrix} P_{11} & P_{12}\end{bmatrix}\mathbf{P}^{-1}\boldsymbol{\alpha}}$$

If $\alpha = \alpha_H = \alpha_L$, we are back to the L96 solution of $V_B = \frac{p}{\alpha}$. ■

A.4 Proofs of Section 4

Recall that debt values are given by

$$\begin{aligned}
\begin{bmatrix} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{bmatrix} &= \mathbf{P}\begin{bmatrix} A_1 + B_1 e^{-\hat{r}_1\tau}[1-F(\delta,\tau)] + C_1 G_1(\delta,\tau) \\ A_2 + B_2 e^{-\hat{r}_2\tau}[1-F(\delta,\tau)] + C_2 G_2(\delta,\tau) \end{bmatrix} \\
&= \mathbf{P}\hat{\mathbf{k}}_0^D + [1-F(\delta,\tau)]\,\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}\mathbf{P}\hat{\mathbf{k}}_F^D + \mathbf{P}\begin{bmatrix} G_1(\delta,\tau) & 0 \\ 0 & G_2(\delta,\tau) \end{bmatrix}\mathbf{P}^{-1}\mathbf{P}\hat{\mathbf{k}}_G^D \\
&= \mathbf{P}\hat{\mathbf{k}}_0^D + [1-F(\delta,\tau)]\exp(-\mathbf{A}\tau)\,\mathbf{P}\hat{\mathbf{k}}_F^D + \mathbf{P}\begin{bmatrix} G_1(\delta,\tau) & 0 \\ 0 & G_2(\delta,\tau) \end{bmatrix}\mathbf{P}^{-1}\mathbf{P}\hat{\mathbf{k}}_G^D
\end{aligned}$$

Here, by defining $a \equiv \frac{\mu - \frac{\sigma^2}{2}}{\sigma^2}$, $\varphi_1 \equiv 0$, $\varphi_2 \equiv -2a$, $\gamma_{i1,2} \equiv -a \pm \frac{\sqrt{a^2\sigma^4 + 2\sigma^2\hat{r}_i}}{\sigma^2}$, and $q(\delta,\rho,t) \equiv \frac{\log(\delta_b) - \log(\delta) - (\rho+a)\cdot\sigma^2 t}{\sigma\sqrt{t}}$, the constants in (8) are given by:

$$\hat{\mathbf{k}}_0^D \equiv \hat{\mathbf{D}}^{-1}\mathbf{P^{-1}c}, \quad \hat{\mathbf{k}}_F^D \equiv p\mathbf{P^{-1}1} - \hat{\mathbf{D}}^{-1}\mathbf{P^{-1}c}, \quad \hat{\mathbf{k}}_G^D \equiv \mathbf{P}^{-1}\boldsymbol{\alpha}\frac{\delta_b}{r-\mu} - \hat{\mathbf{D}}^{-1}\mathbf{P^{-1}c}$$

$$\mathbf{P}\hat{\mathbf{k}}_0^D \equiv \mathbf{A}^{-1}\mathbf{c}, \qquad \mathbf{P}\hat{\mathbf{k}}_F^D \equiv p\mathbf{1} - \mathbf{A}^{-1}\mathbf{c}, \qquad \mathbf{P}\hat{\mathbf{k}}_G^D \equiv \boldsymbol{\alpha}\frac{\delta_b}{r-\mu} - \mathbf{A}^{-1}\mathbf{c}$$

and the functions F and G are given by

$$F(\delta,\tau) \equiv \sum_{i=1}^2 \left(\frac{\delta}{\delta_b}\right)^{\varphi_i} N[q(\delta,\varphi_i,\tau)], \; G_j(\delta,\tau) \equiv \sum_{i=1}^2 \left(\frac{\delta}{\delta_b}\right)^{\gamma_{ji}} N[q(\delta,\gamma_{ji},\tau)],$$

where $N(x)$ is the cumulative distribution function for a standard normal distribution.

Define $\boldsymbol{\omega} \equiv [1,-1]\,\mathbf{A} = \begin{bmatrix} (r_H + \xi_H + \xi_L) \\ -(r_L + \xi_H + \xi_L) \end{bmatrix}^\top$ and $\Pi \equiv D_H - D_L = [1,-1]\begin{bmatrix} D_H \\ D_L \end{bmatrix}$. We will also write the shorthand $\sqrt{\cdot}$ for $\sqrt{[(r+\xi) - (\bar{r} + \lambda\beta)]^2 + 4\xi\lambda\beta}$ and note that $\hat{r}_1 - \sqrt{\cdot} = \hat{r}_2 > 0$.

A.4.1 Time-to-maturity τ derivative

Proof of Proposition 4.

First, we know that at $\tau = 0$, the derivative with respect to τ is

$$\dot{\Pi}(\delta,0) \quad = [1-F(\delta,0)]\,p(r_L - r_H) + \lim_{\tau\to 0}\dot{F}(\delta,0)\frac{\delta_b}{r-\mu}(\alpha_H - \alpha_L) = p(r_L - r_H) > 0$$

and hence our result always holds in the vicinity of $\tau = 0$.

Now we prove the general results under sufficient conditions listed in Proposition 4. First, we note that $q_\tau(\delta, \rho, \tau) = \frac{\log(\delta) - \log(\delta_b) - (\rho + a)\sigma^2 \tau}{\sigma\sqrt{\tau}} \frac{1}{2\tau}$, so δ and δ_b have reversed signs. Then, we have

$$\dot{\left[\begin{array}{c} D_H(\delta, \tau) \\ D_L(\delta, \tau) \end{array}\right]} = \mathbf{P}\left[-\hat{\mathbf{D}} \exp\left(-\hat{\mathbf{D}}\tau\right) \hat{\mathbf{k}}_F^D \left[1 - F(\delta, \tau)\right] - \exp\left(-\hat{\mathbf{D}}\tau\right) \hat{\mathbf{k}}_F^D \dot{F}(\delta, \tau) + \dot{\mathbf{G}}(\delta, \tau) \hat{\mathbf{k}}_G^D\right]$$

and the derivatives of the auxiliary functions are

$$\begin{aligned} \dot{F}(\delta, \tau) &= \sum_{i=1}^{2} \left(\frac{\delta}{\delta_b}\right)^{\varphi_i} \phi\left[q(\delta, \varphi_i, \tau)\right] q_\tau(\delta, \varphi_i, \tau) \\ &= \phi\left[q(\delta, 0, \tau)\right] \sum_{i=1}^{2} q_\tau(\delta, \varphi_i, \tau) \\ &= \phi\left[q(\delta, 0, \tau)\right] \frac{\log\left(\frac{\delta}{\delta_b}\right)}{\sigma\tau^{3/2}} > 0 \\ \dot{G}_j(\delta, \tau) &= \sum_{i=1}^{2} \left(\frac{\delta}{\delta_b}\right)^{\gamma_{ji}} \phi\left[q(\delta, \gamma_{ji}, \tau)\right] q_\tau(\delta, \gamma_{ji}, \tau) \\ &= \phi\left[q(\delta, 0, \tau)\right] e^{-\hat{r}_j \tau} \sum_{i=1}^{2} q_\tau(\delta, \gamma_{ji}, \tau) \\ &= \phi\left[q(\delta, 0, \tau)\right] e^{-\hat{r}_j \tau} \frac{\log\left(\frac{\delta}{\delta_b}\right)}{\sigma\tau^{3/2}} \\ &= e^{-\hat{r}_j \tau} \dot{F}(\delta, \tau) > 0 \end{aligned}$$

where we used

$$\begin{aligned} \left(\frac{\delta}{\delta_b}\right)^{\varphi_i} \phi\left[q(\delta, \varphi_i, \tau)\right] &= \phi\left[q(\delta, 0, \tau)\right] \\ \left(\frac{\delta}{\delta_b}\right)^{\gamma_{ji}} \phi\left[q(\delta, \gamma_{ji}, \tau)\right] &= \phi\left[q(\delta, 0, \tau)\right] e^{-\hat{r}_j \tau} \end{aligned}$$

This is easily derived:

$$\begin{aligned} \left(\frac{\delta}{\delta_b}\right)^{\varphi_i} \phi\left[q(\delta, \varphi_i, \tau)\right] &= e^{-\varphi_i (y_b - y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{y_b - y - (\varphi_i + a)\sigma^2 t}{\sigma\sqrt{t}}\right]^2} \\ &= \exp\left\{-\varphi_i (y_b - y)\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\left[\frac{(y_b - y)^2}{2\sigma^2 t} - 2\frac{(\varphi_i + a)\sigma^2 t (y_b - y)}{2\sigma^2 t} + \frac{\left[(\varphi_i + a)\sigma^2 t\right]^2}{2\sigma^2 t}\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\left[\frac{(y_b - y)^2}{2\sigma^2 t} - 2\frac{a(y_b - y)\sigma^2 t}{2\sigma^2 t} + \frac{(\varphi_i + a)^2 \left(\sigma^2 t\right)^2}{2\sigma^2 t}\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\left[\frac{(y_b - y)^2}{2\sigma^2 t} - 2\frac{a(y_b - y)\sigma^2 t}{2\sigma^2 t} + \frac{a^2 \left(\sigma^2 t\right)^2}{2\sigma^2 t} + \frac{\left(2\varphi_i a + \varphi_i^2\right)\left(\sigma^2 t\right)^2}{2\sigma^2 t}\right]\right\} \\ &= \phi\left[q(\delta, 0, \tau)\right] \exp\left\{-\frac{\left(2\varphi_i a + \varphi_i^2\right)\sigma^2 t}{2}\right\} \end{aligned}$$

and we finally note that $\tilde{\mu}\varphi + \frac{\sigma^2}{2}\varphi^2 = 0 \iff \frac{2\tilde{\mu}}{\sigma^2}\varphi + \varphi^2 = 0 \iff 2\varphi a + \varphi^2 = 0$ which gives the result in conjunction with the fact that $(\varphi_i + a) + (\varphi_{-i} + a) = 0$ as they are complementary roots centered around $-a$. Plugging in, we have

$$\begin{aligned}
\left[\begin{array}{c} \dot{D_H(\delta,\tau)} \\ D_L(\delta,\tau) \end{array}\right] &= \mathbf{P}\left[\begin{array}{c} -\hat{r}_1 B_1 e^{-\hat{r}_1\tau}\left[1-F(\delta,\tau)\right]+(C_1-B_1)e^{-\hat{r}_1\tau}\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2 e^{-\hat{r}_2\tau}\left[1-F(\delta,\tau)\right]+(C_2-B_2)e^{-\hat{r}_2\tau}\dot{F}(\delta,\tau) \end{array}\right] \\
&= \mathbf{P}\left[\begin{array}{cc} e^{-\hat{r}_1\tau} & 0 \\ 0 & e^{-\hat{r}_2\tau} \end{array}\right]\left[\begin{array}{c} -\hat{r}_1 B_1\left[1-F(\delta,\tau)\right]+(C_1-B_1)\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2\left[1-F(\delta,\tau)\right]+(C_2-B_2)\dot{F}(\delta,\tau) \end{array}\right] \\
&= \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\left[\begin{array}{c} -\hat{r}_1 B_1\left[1-F(\delta,\tau)\right]+(C_1-B_1)\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2\left[1-F(\delta,\tau)\right]+(C_2-B_2)\dot{F}(\delta,\tau) \end{array}\right] \\
&= \mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\left(-\left[1-F(\delta,\tau)\right]\hat{\mathbf{D}}\hat{\mathbf{k}}_F^D+\dot{F}(\delta,\tau)\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right) \\
&= \exp(-\mathbf{A}\tau)\left(-\left[1-F(\delta,\tau)\right]\mathbf{AP}\hat{\mathbf{k}}_F^D+\dot{F}(\delta,\tau)\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)
\end{aligned}$$

where we used the fact that $\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)=\exp(-\mathbf{A}\tau)\mathbf{P}$ and $\mathbf{P}\hat{\mathbf{D}}=\mathbf{AP}$. Premultiplying by the difference vector $[1,-1]$ and plugging in the definitions of $\mathbf{A}, B_i, C_i$, we have

$$\begin{aligned}
\dot{\Pi}(\delta,\tau) &= [1,-1]\left[\begin{array}{c} \dot{D_H(\delta,\tau)} \\ D_L(\delta,\tau) \end{array}\right] \\
&= [1,-1]\exp(-\mathbf{A}\tau)\left\{\left[1-F(\delta,\tau)\right]\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]+\dot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_b}{r-\mu}\alpha_H-p \\ \frac{\delta_b}{r-\mu}\alpha_L-p \end{array}\right]\right\}
\end{aligned}$$

Let us derive a formula for a general vector $\left[\begin{array}{c} x \\ y \end{array}\right]$:

$$\begin{aligned}
[1,-1]\exp(-\mathbf{A}\tau)\left[\begin{array}{c} x \\ y \end{array}\right] &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left[x\left(r_L-r_H-\xi_H-\xi_L\right)-y\left(r_H-r_L-\xi_L-\xi_H\right)\right]+\sqrt{\cdot}\left(1+e^{\tau\sqrt{\cdot}}\right)[x-y]\right\} \\
&= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left([r_L,-r_H]\left[\begin{array}{c} x \\ y \end{array}\right]-\boldsymbol{\omega}\left[\begin{array}{c} x \\ y \end{array}\right]\right)+\sqrt{\cdot}\left(1+e^{\tau\sqrt{\cdot}}\right)[1,-1]\left[\begin{array}{c} x \\ y \end{array}\right]\right\} \\
&= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)\left([r_L,-r_H]-\boldsymbol{\omega}+\sqrt{\cdot}[1,-1]\right)\left[\begin{array}{c} x \\ y \end{array}\right]+2\sqrt{\cdot}[1,-1]\left[\begin{array}{c} x \\ y \end{array}\right]\right\}
\end{aligned}$$

When $x>y$, it is clear that for $\tau=0$, we have $[1,-1]\exp(-\mathbf{A}\cdot 0)\left[\begin{array}{c} x \\ y \end{array}\right]=(x-y)>0$. Further, if it is to hold for any τ, we need

$$\left(e^{\tau\sqrt{\cdot}}-1\right)\left([r_L,-r_H]\left[\begin{array}{c} x \\ y \end{array}\right]-\boldsymbol{\omega}\left[\begin{array}{c} x \\ y \end{array}\right]+\sqrt{\cdot}[1,-1]\left[\begin{array}{c} x \\ y \end{array}\right]\right)\geq 0$$

Our derivation of $\dot{\Pi}$ has two terms of this form, multiplied by $[1-F]>0$ and $\dot{F}>0$. To ensure positivity, this implies conditions on $p, c, r_H, r_L, \alpha_H, \alpha_L, \delta_b$ once we identify $\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]$ and $\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} \frac{\delta_b}{r-\mu}\alpha_H-p \\ \frac{\delta_b}{r-\mu}\alpha_L-p \end{array}\right]$. Define$V_B\equiv\frac{\delta_b}{r-\mu}$; thus, we have the following two conditions for these two cases, i.e., for $\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]$,

$$-\left(r_L-r_H\right)\left[p\left(r_H+r_L+\xi_H+\xi_L-\sqrt{\cdot}\right)-2c\right] \quad > \quad \Longleftrightarrow w_1\equiv c-p\hat{r}_2>0 \Longleftrightarrow \left(r_L-r_H\right)2\left[c-p\hat{r}_2\right]>0$$

and for $\left[\begin{array}{c} V_B\alpha_H-p \\ V_B\alpha_L-p \end{array}\right]$,

$$\begin{aligned}
V_B\left[\alpha_L\left(r_L-r_H+\xi_H+\xi_L-\sqrt{\cdot}\right)-\alpha_H\left(r_H-r_L+\xi_H+\xi_L-\sqrt{\cdot}\right)\right]+2p\left(r_H-r_L\right) &> 0 \\
\Longleftrightarrow V_B\left[\alpha_L\left(-2r_H+2\hat{r}_2\right)-\alpha_H\left(-2r_L+2\hat{r}_2\right)\right]+2p\left(r_H-r_L\right) &> 0 \\
\Longleftrightarrow w_2\equiv V_B\left[\alpha_L\left(\hat{r}_2-r_H\right)+\alpha_H\left(r_L-\hat{r}_2\right)\right]-p\left(r_L-r_H\right) &> 0
\end{aligned}$$

Note that $r_H<\hat{r}_2<r_L$. So we need sufficiently high $c>p\hat{r}_2$ and also sufficiently high α_L, α_H in the face of a large discount differential r_L-r_H. We thus have proved the following proposition. Thus, under the sufficient conditions listed in Proposition 4, we have

$$\begin{aligned} w_1 \equiv c - p\hat{r}_2 &\geq 0 \\ w_2 \equiv V_B\left[\alpha_L\left(\hat{r}_2 - r_H\right) + \alpha_H\left(r_L - \hat{r}_2\right)\right] - p\left(r_L - r_H\right) &\geq 0, \end{aligned}$$

which implies that $\Pi_\tau(\delta,\tau) > 0$, i.e. the bid-ask spread $(1-\beta)\,\Pi(\delta,\tau)$ is larger for bonds with longer time-to-maturity. ■

A.4.2 Proof of $\Pi' < 0$ via the system of PDEs and LHS

Proposition 7 *Under sufficient conditions listed below, we have $\Pi_\delta(\delta,\tau) < 0$, i.e. the bid-ask spread is smaller for bonds with higher firm fundamental.*

Proof of Proposition 5.
We aim to prove $\Pi' < 0$ under the following sufficient conditions:

$$\frac{\delta_b}{r-\mu}\left(\alpha_H - \alpha_L\right) - p\left(r_L - r_H\right) > 0 \quad \text{(A.1)}$$

$$-\alpha_L\left(r_L - r_H\right) + \left(\alpha_H - \alpha_L\right)\frac{r_H + \xi_H + \xi_L}{2} > 0 \quad \text{(A.2)}$$

$$\frac{\delta_b}{r-\mu}\left(\alpha\left(r_L - r_H\right)\hat{r}_1 + \left(\alpha_H - \alpha_L\right)\frac{\left[\left(r_H + \xi_H\right)\left(r_L - r_H - \xi_H\right) - \xi_L\left(r_L + \xi_L + 2\xi_H\right)\right]}{2\sqrt{\cdot}}\right) - c\frac{\left(r_L - r_H\right)}{\sqrt{\cdot}} > 0 \quad \text{(A.3)}$$

$$\left(\alpha_H - \alpha_L\right)\frac{\delta_b}{r-\mu} - \left(r_L - r_H\right)\frac{c}{\left[\left(r_H + \xi_H\right)\left(r_L + \xi_L\right) - \xi_H\xi_L\right]} > 0 \quad \text{(A.4)}$$

First, note that when we subtract the second line from the first line of the differential equation we have

$$\begin{aligned} [1,-1]\begin{bmatrix} r_H + \xi_H & -\xi_H \\ -\xi_L & r_L + \xi_L \end{bmatrix}\begin{bmatrix} D_H \\ D_L \end{bmatrix} &= [1,-1]\left(\begin{bmatrix} c \\ c \end{bmatrix} + \tilde{\mu}\delta\begin{bmatrix} D_H \\ D_L \end{bmatrix}' + \frac{\sigma^2}{2}\delta^2\begin{bmatrix} D_H \\ D_L \end{bmatrix}'' - \begin{bmatrix} \dot{D}_H \\ D_L \end{bmatrix}\right) \\ \iff \boldsymbol{\omega}\begin{bmatrix} D_H \\ D_L \end{bmatrix} + \dot{\Pi} &= \tilde{\mu}\Pi' + \frac{\sigma^2}{2}\Pi'' \\ \iff LHS &= \tilde{\mu}\Pi' + \frac{\sigma^2}{2}\Pi'' \end{aligned}$$

where

$$\boldsymbol{\omega} \equiv \left[r_H + \xi_H + \xi_L, -\left(r_L + \xi_L + \xi_H\right)\right].$$

Let us first establish a limit of $LHS(\delta,\tau)$:

$$\lim_{\tau\to 0} LHS(\delta,\tau) = \boldsymbol{\omega}\begin{bmatrix} D_H(\delta,0) \\ D_L(\delta,0) \end{bmatrix} + \lim_{\tau\to 0}\dot{\Pi}(\delta,\tau) = -p\left(r_L - r_H\right) + p\left(r_L - r_H\right) = 0.$$

Outline of the proof:

1. Show that $L\dot{H}S$ as a function of τ only changes sign once.
2. Show, when τ is small, that LHS increases, that is

$$L\dot{H}S(\delta,\tau) > 0.$$

3. Show that $LHS(\delta,\infty) \geq 0$.
4. Show that

$$\Pi(\delta_b,\tau) - \lim_{\delta\to\infty}\Pi(\delta,\tau) > 0$$

Then we are done: (1.) implies that the can at most be one local extrema. By (2.), we know that there is a local maximum in LHS in terms of τ, i.e., LHS has to go up and then down again to approach from above the value in (3.), which is zero or something positive. Finally, (4.) gives us a contradiction if ever $\Pi' > 0$. First, by continuity of the expectation, we have that $\Pi' < 0$ for some part of the state space (δ_b, ∞), as otherwise the surplus couldn't be less at ∞ than at 0. Suppose now that there is an interval on which $\Pi' < 0$. This means that there exist a local maximum with $\Pi' = 0 > \Pi''$. But this would imply $LHS = \tilde{\mu}\Pi' + \frac{\sigma^2}{2}\Pi'' < 0$, a contradiction. Thus, $\Pi' > 0$ everywhere.

Step 1: Recall that

$$\left[\begin{array}{c} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right]^{\cdot} = \exp(-\mathbf{A}\tau)\mathbf{P}\left[\begin{array}{c} -\hat{r}_1 B_1[1-F(\delta,\tau)]+(C_1-B_1)\dot{F}(\delta,\tau) \\ -\hat{r}_2 B_2[1-F(\delta,\tau)]+(C_2-B_2)\dot{F}(\delta,\tau) \end{array}\right]$$
$$= \exp(-\mathbf{A}\tau)\mathbf{P}\left(-[1-F(\delta,\tau)]\hat{\mathbf{D}}\hat{\mathbf{k}}_F^D+\dot{F}(\delta,\tau)\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)$$

Thus, we have

$$\left[\begin{array}{c} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right]^{\cdot\cdot} = \exp(-\mathbf{A}\tau)(-\mathbf{A})\mathbf{P}\left(-[1-F(\delta,\tau)]\hat{\mathbf{D}}\hat{\mathbf{k}}_F^D+\dot{F}(\delta,\tau)\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)$$
$$+\exp(-\mathbf{A}\tau)\mathbf{P}\left(\dot{F}(\delta,\tau)\hat{\mathbf{D}}\hat{\mathbf{k}}_F^D+\ddot{F}(\delta,\tau)\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)$$
$$= \exp(-\mathbf{A}\tau)\left([1-F(\delta,\tau)]\mathbf{A}^2\mathbf{P}\hat{\mathbf{k}}_F^D-\dot{F}(\delta,\tau)\mathbf{A}\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)$$
$$+\exp(-\mathbf{A}\tau)\left(\dot{F}(\delta,\tau)\mathbf{A}\mathbf{P}\hat{\mathbf{k}}_F^D+\dot{F}(\delta,\tau)(\ldots)\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)$$
$$= \exp(-\mathbf{A}\tau)\left([1-F(\delta,\tau)]\mathbf{A}^2\mathbf{P}\hat{\mathbf{k}}_F^D\right)$$
$$+\exp(-\mathbf{A}\tau)\dot{F}(\delta,\tau)\left(\mathbf{A}\mathbf{P}\left[\begin{array}{c} 2B_1-C_1 \\ 2B_2-C_2 \end{array}\right]+(\ldots)\mathbf{I}\mathbf{P}\left[\begin{array}{c} C_1-B_1 \\ C_2-B_2 \end{array}\right]\right)$$

where we used the fact that $\mathbf{AP}=\mathbf{P}\hat{\mathbf{D}}$ and $\mathbf{A}\exp(-\mathbf{A}\tau)=\mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\hat{\mathbf{D}}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\hat{\mathbf{D}}\mathbf{P}^{-1}=\exp(-\mathbf{A}\tau)\mathbf{A}$ as diagonal matrices of the same order commute.

Thus, if we can show that $L\dot{H}S>0$ for any $\delta>\delta_b$ we are done. Note that $\frac{\partial^2\Pi(\delta,\tau)}{\partial\tau^2}=\ddot{\Pi}$ equals to

$$\ddot{\Pi} = [1,-1](-\mathbf{A})\left[\begin{array}{c} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right]^{\cdot}+[1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]+\ddot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_b}{r-\mu}\alpha_H-p \\ \frac{\delta_b}{r-\mu}\alpha_L-p \end{array}\right]\right\}$$
$$= [1,-1]\exp(-\mathbf{A}\tau)\left\{-\mathbf{A}\left([1-F(\delta,\tau)]\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]+\dot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_b\alpha_H}{r-\mu}-p \\ \frac{\delta_b\alpha_L}{r-\mu}-p \end{array}\right]\right)-\dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]+\ddot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_b\alpha_H}{r-\mu}-p \\ \frac{\delta_b\alpha_L}{r-\mu}-p \end{array}\right]\right\}$$
$$= -\boldsymbol{\omega}\left[\begin{array}{c} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right]^{\cdot}+[1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]+\ddot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_b}{r-\mu}\alpha_H-p \\ \frac{\delta_b}{r-\mu}\alpha_L-p \end{array}\right]\right\}$$

where we used the fat that
$\mathbf{A}\exp(-\mathbf{A}\tau)=\mathbf{P}\hat{\mathbf{D}}\mathbf{P}^{-1}\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\hat{\mathbf{D}}\exp\left(-\hat{\mathbf{D}}\tau\right)\mathbf{P}^{-1}=\mathbf{P}\exp\left(-\hat{\mathbf{D}}\tau\right)\hat{\mathbf{D}}\mathbf{P}^{-1}=\exp(-\mathbf{A}\tau)\mathbf{A}$ as diagonal matrices of the same order commute.

We realize that the $\boldsymbol{\omega}\left[\begin{array}{c} D_H(\delta,\tau) \\ D_L(\delta,\tau) \end{array}\right]^{\cdot}$ parts cancel out in $L\dot{H}S$, and we are left with

$$L\dot{H}S(\delta,\tau)=[1,-1]\exp(-\mathbf{A}\tau)\left\{-\dot{F}(\delta,\tau)\left[\begin{array}{c} c-pr_H \\ c-pr_L \end{array}\right]+\ddot{F}(\delta,\tau)\left[\begin{array}{c} \frac{\delta_b}{r-\mu}\alpha_H-p \\ \frac{\delta_b}{r-\mu}\alpha_L-p \end{array}\right]\right\}$$

Further note that with $\dot{F}(\delta,\tau)=\phi\left[q(\delta,0,\tau)\right]\frac{\log\left(\frac{\delta}{\delta_b}\right)}{\sigma\tau^{3/2}}$, $q_\tau(\delta,0,\tau)=\frac{\log\left(\frac{\delta}{\delta_b}\right)-a\sigma^2\tau}{2\sigma\tau^{3/2}}$, and $\phi'(x)=-x\phi(x)$, we have

$$\begin{aligned}
\ddot{F}(\delta,\tau) &= \phi'\left[q(\delta,0,\tau)\right]q_\tau(\delta,0,\tau)\frac{\log\left(\frac{\delta}{\delta_b}\right)}{\sigma\tau^{3/2}}+\phi\left[q(\delta,0,\tau)\right]\frac{\log\left(\frac{\delta}{\delta_b}\right)}{\sigma\tau^{3/2}}\left(-\frac{3}{2\tau}\right)\\
&= \dot{F}(\delta,\tau)\left[-q(\delta,0,\tau)\,q_\tau(\delta,0,\tau)-\frac{3}{2\tau}\right]\\
&= \dot{F}(\delta,\tau)\left[-\frac{-\log\left(\frac{\delta}{\delta_b}\right)-a\sigma^2\tau}{\sigma\sqrt{\tau}}\cdot\frac{\log\left(\frac{\delta}{\delta_b}\right)-a\sigma^2\tau}{2\sigma\tau^{3/2}}-\frac{3}{2\tau}\right]\\
&= \dot{F}(\delta,\tau)\left[\frac{\log\left(\frac{\delta}{\delta_b}\right)^2-a^2\left(\sigma^2\right)^2\tau^2}{2\sigma^2\tau^2}-\frac{3}{2\tau}\right]\\
&= \dot{F}(\delta,\tau)\left[\frac{\log\left(\frac{\delta}{\delta_b}\right)^2}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right]
\end{aligned}$$

so that

$$L\dot{H}S(\delta,\tau)=\dot{F}(\delta,\tau)\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\left\{\left(\frac{\log\left(\frac{\delta}{\delta_b}\right)^2}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left[\begin{array}{c}\frac{\delta_b}{r-\mu}\alpha_H-p\\ \frac{\delta_b}{r-\mu}\alpha_L-p\end{array}\right]-\left[\begin{array}{c}c-pr_H\\ c-pr_L\end{array}\right]\right\}$$

Let us now write out this term in more detail. First, note that

$$\begin{aligned}
\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\left[\begin{array}{c}V_B\alpha_H-p\\ V_B\alpha_L-p\end{array}\right] &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}\\
\left[1,-1\right]\exp\left(-\mathbf{A}\tau\right)\left[\begin{array}{c}c-pr_H\\ c-pr_L\end{array}\right] &= \frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\times\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}
\end{aligned}$$

Then, let $x\equiv\log\left(\frac{\delta}{\delta_b}\right)^2\in(0,\infty)$, to simplify to

$$L\dot{H}S=\dot{F}\times\frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}-\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}\right]$$

As $\dot{F}\times\frac{e^{-\hat{r}_1\tau}}{2\sqrt{\cdot}}>0$, we know that the term $[\cdot]$ determines the sign of $L\dot{H}S$. Writing it out, we have

$$\begin{aligned}
&\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2+2\sqrt{\cdot}V_B\left(\alpha_H-\alpha_L\right)\right\}-\left\{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1+2\sqrt{\cdot}p\left(r_L-r_H\right)\right\}\right]\\
=\;&\left(e^{\tau\sqrt{\cdot}}-1\right)\left[\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)w_2-w_1\right]+2\sqrt{\cdot}\left[V_B\left(\alpha_H-\alpha_L\right)-p\left(r_L-r_H\right)\right]
\end{aligned}$$

We note that $\lim_{\tau\to0}\frac{e^{\tau\sqrt{\cdot}}-1}{\tau}=\frac{"0"}{"0"}=\sqrt{\cdot}>0$, so that $\lim_{\tau\to\infty}\frac{e^{\tau\sqrt{\cdot}}-1}{\tau^2}=\infty$. Thus, at τ in the vicinity of 0, the sign of the term is determined by w_2. Next, when $\tau\to\infty$, we have the sign being determined by $-\frac{a^2\sigma^2}{2}w_2-w_1<0$.

Multiplying out $w_2\left(e^{\tau\sqrt{\cdot}}-1\right)>0$, and defining $Q_1(x,\tau)=\left(\frac{x}{\sigma^2\tau^2}-\frac{a^2\sigma^2}{2}-\frac{3}{2\tau}\right)$, we have

$$\begin{aligned}
Q(x,\tau) &= Q_1(x,\tau)-\frac{w_1}{w_2}+\frac{2\sqrt{\cdot}\left[V_B(\alpha_H-\alpha_L)-p(r_L-r_H)\right]}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\
&= Q_1(x,\tau)-\frac{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-2\sqrt{\cdot}\left[V_B(\alpha_H-\alpha_L)-p(r_L-r_H)\right]}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\
&= Q_1(x,\tau)-\frac{\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-w_3}{\left(e^{\tau\sqrt{\cdot}}-1\right)w_2}\\
&= Q_2(x,\tau)-Q_2(\tau)
\end{aligned}$$

where from (A.1) we know that

$$w_3\equiv 2\sqrt{\cdot}\left[V_B(\alpha_H-\alpha_L)-p(r_L-r_H)\right]>0.$$

Note that $Q_1(x,\tau)$ changes sign only once. Then, we know that

$$\dot{Q}_2(\tau) = \frac{\sqrt{\cdot}e^{\tau\sqrt{\cdot}}w_1\left(e^{\tau\sqrt{\cdot}}-1\right)w_2-\left[\left(e^{\tau\sqrt{\cdot}}-1\right)w_1-w_3\right]\sqrt{\cdot}e^{\tau\sqrt{\cdot}}w_2}{(\cdot)^2}=\frac{w_2w_3\sqrt{\cdot}e^{\tau\sqrt{\cdot}}}{(\cdot)^2}$$

Thus, if $w_2w_3>0$, then $\dot{Q}_2(\tau)>0$ and we know that $Q(x,\tau)$ is composed of a part that crosses from positive to negative as τ increase ($Q_1(x,\tau)$) and of a part that is monotonically decreasing as τ increases ($-Q_2(\tau)$).

Step 2: From the derivation above, we know that for τ in the vicinity of 0, the sign of the $L\dot{H}S$ is determined by w_2. Next, when $\tau\to\infty$, we have the sign being determined by $-\frac{a^2\sigma^2}{2}w_2-w_1<0$.

Step 3: Note that

$$LHS(\delta,\infty)=\boldsymbol{\omega}\mathbf{P}\begin{bmatrix}\left(\frac{\delta}{\delta_b}\right)^{\gamma_{12}} & 0\\ 0 & \left(\frac{\delta}{\delta_b}\right)^{\gamma_{22}}\end{bmatrix}\mathbf{P}^{-1}\mathbf{P}\hat{\mathbf{k}}_G^D$$

with $\gamma_{12}<\gamma_{22}<0$, so that $0<X_1=\left(\frac{\delta}{\delta_b}\right)^{\gamma_{12}}<\left(\frac{\delta}{\delta_b}\right)^{\gamma_{22}}=X_2$. Note that for $\delta\to\delta_b$, the LHS is positive under (A.2)

$$\lim_{\delta\to\delta_b}LHS(\delta,\infty)=-\alpha_L(r_L-r_H)+(\alpha_H-\alpha_L)\frac{r_H+\xi_H+\xi_L}{2}>0.$$

First, let us note the following results:

$$\begin{aligned}
\mathbf{P}\hat{\mathbf{k}}_G^D &= \boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1\frac{\delta_b}{r-\mu}-\mathbf{A}^{-1}[XXX]\\
\boldsymbol{\omega}\mathbf{P}\begin{bmatrix}X_1 & 0\\ 0 & X_2\end{bmatrix}\mathbf{P}^{-1}\boldsymbol{\alpha} &= \boldsymbol{\omega}\mathbf{P}\begin{bmatrix}X_1 & 0\\ 0 & X_2\end{bmatrix}\mathbf{P}^{-1}\begin{bmatrix}\alpha_H\\ \alpha_L\end{bmatrix}\\
&= \frac{\alpha_L(r_L-r_H)\left[\hat{r}_1(X_2-X_1)-X_2\sqrt{\cdot}\right]}{\sqrt{\cdot}}\\
&\quad +(\alpha_H-\alpha_L)(X_2-X_1)\frac{\left[(r_H+\xi_H)(r_L-r_H-\xi_H)-\xi_L(r_L+\xi_L+2\xi_H)\right]}{2\sqrt{\cdot}}\\
&\quad +(\alpha_H-\alpha_L)(X_1+X_2)\frac{r_H+\xi_H+\xi_L}{2}\\
\boldsymbol{\omega}\mathbf{P}\begin{bmatrix}X_1 & 0\\ 0 & X_2\end{bmatrix}\mathbf{P}^{-1}\mathbf{A}^{-1}\mathbf{1} &= \frac{(r_L-r_H)(X_2-X_1)}{\sqrt{\cdot}}>0
\end{aligned}$$

Combining these results, we have that

$$
\begin{aligned}
& LHS\left(\delta,\infty\right) \\
= \quad & \frac{\delta_b}{r-\mu}\times\left\{\frac{\alpha_L\left(r_L-r_H\right)\left[\hat{r}_1\left(X_2-X_1\right)-X_2\sqrt{\cdot}\right]}{\sqrt{\cdot}}+\left(\alpha_H-\alpha_L\right)\left(X_2-X_1\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\right\} \\
& +\frac{\delta_b}{r-\mu}\times\left\{\left(\alpha_H-\alpha_L\right)\left(X_1+X_2\right)\frac{r_H+\xi_H+\xi_L}{2}\right\}-c\frac{\left(r_L-r_H\right)\left(X_2-X_1\right)}{\sqrt{\cdot}} \\
= \quad & \left(X_2-X_1\right)\left[\frac{\delta_b}{r-\mu}\left(\alpha\left(r_L-r_H\right)\hat{r}_1+\left(\alpha_H-\alpha_L\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\right)-c\frac{\left(r_L-r_H\right)}{\sqrt{\cdot}}\right] \\
& +X_2\frac{\delta_b}{r-\mu}\left(\frac{r_H+\xi_H+\xi_L}{2}\left(\alpha_H-\alpha_L\right)-\alpha_L\left(r_L-r_H\right)\right)+X_1\frac{\delta_b}{r-\mu}\frac{r_H+\xi_H+\xi_L}{2}\left(\alpha_H-\alpha_L\right)
\end{aligned}
$$

Because $0<X_1<X_2$, the sufficient conditions for $LHS\left(\delta,\infty\right)>0$ are

$$
\begin{aligned}
\frac{\delta_b}{r-\mu}\left(\alpha\left(r_L-r_H\right)\hat{r}_1+\left(\alpha_H-\alpha_L\right)\frac{\left[\left(r_H+\xi_H\right)\left(r_L-r_H-\xi_H\right)-\xi_L\left(r_L+\xi_L+2\xi_H\right)\right]}{2\sqrt{\cdot}}\right)-c\frac{\left(r_L-r_H\right)}{\sqrt{\cdot}} &> 0, \\
\frac{r_H+\xi_H+\xi_L}{2}\left(\alpha_H-\alpha_L\right)-\alpha_L\left(r_L-r_H\right) &> 0.
\end{aligned}
$$

Here, the first condition is from (A.3) and the second is from (A.2).

Step 4: We have

$$
\begin{aligned}
\Pi\left(\delta_b,\tau\right) &= \frac{\delta_b}{r-\mu}\left(\alpha_H-\alpha_L\right) \\
\lim_{\delta\to\infty}\Pi\left(\delta,\tau\right) &= [1,-1]\left[\mathbf{A}^{-1}\left[XXX\right]+\exp\left(-\mathbf{A}\tau\right)\left(p\mathbf{1}-\mathbf{A}^{-1}\left[XXX\right]\right)\right]
\end{aligned}
$$

Under our assumption that $\Pi_\tau\left(\delta,\tau\right)>0$, we know that the highest $\Pi\left(\delta,\tau\right)$ is at $\tau=\infty$. Noting

$$
\begin{aligned}
[1,-1]\,\mathbf{A}^{-1}\mathbf{1} &= \frac{r_L-r_H}{\left(r_H+\xi_H\right)\left(r_L+\xi_L\right)-\xi_H\xi_L} \\
[1,-1]\exp\left(-\mathbf{A}\tau\right)\mathbf{1} &= \frac{\left(r_L-r_H\right)e^{-\hat{r}_1\tau}\left(e^{\sqrt{\cdot}\tau}-1\right)}{\sqrt{\cdot}} \\
[1,-1]\exp\left(-\mathbf{A}\tau\right)\mathbf{A}^{-1}\mathbf{1} &= -\frac{\left(r_L-r_H\right)e^{-\hat{r}_1\tau}\left[\hat{r}_1\left(e^{\sqrt{\cdot}\tau}-1\right)+\sqrt{\cdot}\right]}{\sqrt{\cdot}\left[\left(r_H+\xi_H\right)\left(r_L+\xi_L\right)-\xi_H\xi_L\right]}
\end{aligned}
$$

we have from (A.4)

$$
\begin{aligned}
\Pi\left(\delta_b,\tau\right)-\lim_{\delta\to\infty}\Pi\left(\delta,\tau\right) &> \lim_{\tau\to\infty}\left\{\Pi\left(\delta_b,\tau\right)-\lim_{\delta\to\infty}\Pi\left(\delta,\tau\right)\right\} \\
&= \left(\alpha_H-\alpha_L\right)\frac{\delta_b}{r-\mu}-\left(r_L-r_H\right)\frac{c}{\left[\left(r_H+\xi_H\right)\left(r_L+\xi_L\right)-\xi_H\xi_L\right]}>0.
\end{aligned}
$$

Taken together, we established parameter restrictions that result in $\Pi_\delta\left(\delta,\tau\right)<0$. ■

Looser sufficiency conditions can be established for $\Pi_\delta\left(\delta,\tau\right)$ in the vicinity of $\tau=0$ or $\delta=\delta_b$. We omit these proofs for brevity.

A.5 The steady-state distribution of types, trading volume

We now derive the cross-sectional (w.r.t. τ) steady-state distribution of L types. Let $p_{HS}\left(t,\tau\right)$ be the proportion at time t of H and S types of maturity τ. Then we have

$$
\frac{\partial p_{HS}\left(t,\tau\right)}{\partial t}-\frac{\partial p_{HS}\left(t,\tau\right)}{\partial\tau}=\lambda p_L\left(t,\tau\right)-\xi p_{HS}\left(t,\tau\right)
$$

as when time advances, maturity shrinks. To impose a steady-state, we note that $\frac{\partial p_{HS}(t,\tau)}{\partial t}=0$ and that $p_{HS}\left(t,T\right)=1$, i.e., at any time t, due to the firm being able to issue to only H types, the proportion of H types with the longest maturity T is always 1. Further note that $p_{HS} + p_L = 1, \forall \tau$, so that in the end we have

$$
\begin{aligned}
-\frac{\partial p_{HS}(\tau)}{\partial \tau} &= \lambda p_L(t,\tau) - \xi p_{HS}(t,\tau) \\
p_{HS}(\tau) &= \frac{\lambda + \xi e^{(\tau - T)(\lambda+\xi)}}{\lambda+\xi} \\
p_L(\tau) &= \frac{\xi}{\lambda+\xi}\left[1 - e^{(\tau-T)(\lambda+\xi)}\right]
\end{aligned}
$$

We of course have to adjust by the density of bonds $\frac{1}{T}$ when looking at the steady state mass of H and L types, μ_{HS} and μ_L. We solve to get

$$
\begin{aligned}
\mu_{HS}(T) &= \frac{1}{T}\int_0^T p_{HS}(\tau)\, d\tau \\
&= \frac{\lambda}{\lambda+\xi} + \frac{\xi\left(1 - e^{-T(\lambda+\xi)}\right)}{T(\lambda+\xi)^2} \\
\mu_L(T) &= \frac{\xi}{\lambda+\xi} - \frac{\xi\left(1 - e^{-T(\lambda+\xi)}\right)}{T(\lambda+\xi)^2}
\end{aligned}
$$

and we note that $\mu'_L(T) > 0 > \mu'_{HS}(T)$ (note that $\frac{\partial p_i(\tau)}{\partial T} \neq 0$), $\lim_{T\to 0}\mu_{HS}(T) = 1$ and $\lim_{T\to 0}\mu_L(T) = 0$, as well as $\lim_{T\to\infty}\mu_{HS}(T) = \frac{\lambda}{\lambda+\xi}$ and $\lim_{T\to\infty}\mu_L(T) = \frac{\xi}{\lambda+\xi}$.

Trade volume is now easily derived. It is simply the mass of agents that are in state (L,τ) times the intensity with which they meet a market maker and execute trades, λ. Thus, trade volume (scaled by total bonds outstanding) for maturity τ will be

$$
Volume(\tau) = \frac{\lambda}{T} p_L(\tau) = \frac{1}{T}\frac{\lambda\xi}{\lambda+\xi}\left[1 - e^{(\tau-T)(\lambda+\xi)}\right]
$$

A.6 Steady state in the search market

So far, we have assumed that the intermediation intensity λ is exogenously determined by the dealers. This assumption hinges on the fact that we assume an infinite mass of H type buyers waiting on the sideline who do not hold the asset, but in order to buy have to go through a dealer. We can relax these assumptions in several ways without substantially changing the model.

However, the two assumptions we will not relax are that (i) orders are 'batched' in the sense that there is no difference in intermediation intensities between different maturities $\tau \in [0,T]$, and (ii) that dealers extract all the surplus when bargaining with H type buyers. Relaxing (i) would destroy our closed form solution without providing much more insight.

The problem with relaxing (ii) is more subtle: by removing dealers and allowing direct negotiation between H and L types we essentially (except in extreme circumstances) leave some surplus beyond their own valuation to the H type buyers. If we assume that we have an infinite mass of H type buyers on the sideline, this would not change their behavior as every single one of the buyers expects never to have the opportunity to be able to buy and make a surplus. However, if there is only a finite mass of H type buyers waiting on the sideline (as we will allow later on), then (a) the firm's pricing at issuance will be affected as it will have to entice H type buyers to participate instead of waiting and buying for a discount in the secondary market, (b) we now have to track the value function of the H and L types who do not hold the asset, and (c) the expected surplus for an H type not holding the asset is an integral over all possible maturities he might encounter. This sufficiently complicates the equations to render them not solvable by the methods we employed in this paper, while not adding more insights. In a follow up project, we can easily incorporate a dealer-less market by giving up the deterministic maturity dimension.

First, we can easily relax the model to allow for random transitioning back from the L to the H state for bondholders, say with intensity ζ. Then we can simply use our current valuation formulas, but with $\lambda\beta + \zeta$ taking the place of $\lambda\beta$. Also, the impact on trade-volume can be easily handled but for brevity is not shown here.

Second, with this switching back intensity, we can also close the model to have 4 different finite population measures — H types with and without the bond, and L types with and without the bond under the assumption that a dealer will only intermediate a trade once he found a buying party (H type) and leave no surplus to the H type buyer. This then allows us, in the tradition of most search models, to define the meeting intensity λ as some function of the steady-state masses of these populations, especially of the mass H types without the bond trying to buy and the mass of L types with the bond trying to sell. This would, in steady-state, result in a meeting intensity $\lambda(T)$ that is a function of the maturity structure, which however for a given T would be constant. The valuation equations would simply include $\lambda(T)$ as a constant and thus would not change. The only thing that would change is the optimal maturity calculations we analyzed in Section 5.1 — here, the firm will take into account the impact of its maturity choice T on the liquidity of the secondary market, and thus indirectly on the valuation of its bonds.