

Learning whether other Traders are Informed*

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Abstract

In standard rational expectations models, investors know whether others are informed, and therefore *know* how to update their beliefs using prices. We develop a dynamic model in which investors must learn whether others are informed and, therefore, *learn* how to use the information in prices. We show that the price is a non-linear function of the underlying signal, and expected returns and volatility are stochastic and persistent, even though shocks to fundamentals and signals are i.i.d. The price reaction to information about dividends is asymmetric: the price reacts more strongly to bad news than it does to good news. The model also generates volatility clustering in which large return realizations, which are associated with dividend surprises, are followed by higher volatility and higher expected returns.

JEL Classification: G12, G14

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1 Introduction

Standard rational expectations (RE) models assume that all participants know whether other agents in the market are informed. This implies that all agents, including those who are uninformed about fundamentals, understand how informative prices are about fundamentals.¹ Arguably, this requires an unrealistic degree of sophistication on the part of uninformed investors. That is, it seems unlikely that investors who are *uncertain* about fundamentals, know with *certainty* whether other investors are privately informed.

In this paper, we develop a framework to study return dynamics when investors must *learn* whether others are informed. Investors have mean-variance preferences and trade competitively in a centralized market by submitting limit orders. There are two groups of investors: the *uninformed* and the *potentially informed*. Potentially informed investors receive a signal in each period which they believe is informative about next period's dividends. Uninformed investors use prices and realized dividends to learn about whether the potentially informed investors are actually informed and, consequently, to update their beliefs about future dividends. As such, the uninformed investors in our model *learn* about whether the price is informative about fundamentals.²

The model predicts return dynamics that are consistent with empirical evidence but are difficult to generate in standard RE models. This is because, in our model, the price is non-linear in the aggregate information and depends on the uninformed investors' beliefs about whether other investors are informed. Since these beliefs evolve endogenously over time, expected returns and return volatility are both stochastic and persistent, even though shocks to fundamentals and signals are i.i.d. The model predicts that the price reacts more strongly to negative news than it does to positive news. In fact, the price may even decrease following positive signals about dividends. Moreover, the model can generate volatility clustering in which large (positive or negative), unexpected return realizations in the current period are followed by higher return volatility and higher expected returns in the next period. Finally, the relation between information quality and returns endogenously varies over time and depends on the degree to which investors agree on the informativeness of the signal: perhaps surprisingly, higher quality information can lead to higher expected returns and volatility when there is sufficient disagreement.

The mechanism driving these results derives from the uninformed investors' filtering prob-

¹For instance, in Kyle (1985), the uninformed market maker knows that there exists an informed strategic trader as well as the precision of that trader's private information. Similarly, in Grossman and Stiglitz (1980), uninformed investors know the number of informed investors in the market and the precision of their signals.

²In our setting, the presence of noise traders is unnecessary for generating non-trivial outcomes. Although the signal of the potentially informed is perfectly revealed in equilibrium, uninformed investors are unsure whether the signal conveys payoff relevant information and trade occurs due to the lack of a common prior.

lem. Conditional on the other traders being informed, an uninformed investor's inference problem is linear in the signal — a positive signal (i.e., good news) increases the conditional expectation about fundamentals, and a negative signal (i.e., bad news) decreases the conditional expectation. However, since uninformed investors are unsure about whether informed traders are present, a surprise in the signal (in either direction) increases the uninformed investors' uncertainty, or posterior variance, about fundamentals. As a result, the posterior variance of uninformed investors depends on the signal, which, in turn, generates stochastic expected returns and return volatility.

Prices react asymmetrically to news due to the multi-dimensionality of the filtering problem. When there is a negative surprise, an uninformed investor's conditional expectation is lower and her conditional variance is higher, both of which lead to a decrease in the price. However, when there is a positive surprise, the conditional expectation is higher but so is the conditional variance, and these have off-setting effects on the price. As a result, prices are more sensitive to bad news, or negative surprises, than to good news. When the magnitude of a positive surprise in the signal is small relative to the overall risk concerns, the effect on the conditional variance dominates and the price *decreases* with additional good news.

Volatility clustering is a result of how an uninformed investor updates her beliefs in response to realized dividends. Since an uninformed investor forms her conditional expectation of next period's dividends based on the signal of the potentially informed investors, a dividend realization that is far from her conditional expectation (i.e., a large dividend surprise) leads her to revise her belief about the informativeness of the signal downwards. In other words, big surprises in dividend realizations, which are accompanied by return realizations, reduce the likelihood that the potentially informed investors are actually informed. In turn, this can increase the uninformed investor's uncertainty about fundamentals and, therefore, lead to higher volatility and higher expected returns in future periods.

Finally, we show that the relation between the information quality of the signal and return moments depends on whether investors agree on the interpretation of the signal. For instance, if investors agree on informativeness of the signal (i.e., if the uninformed investors put a high probability on the other traders being informed), then higher information quality reduces uncertainty about fundamentals and, intuitively, leads to lower expected returns and return volatility. However, if the uninformed investors believe that other investors are not likely to be informed, the opposite relation obtains. A more informative signal for the potentially informed investors induces them to trade more aggressively, which, from an uninformed investor's perspective, introduces more noise to current and future prices and, therefore, leads to higher expected returns and volatility. Since the uninformed investors' beliefs about whether the others are informed evolves over time, the relation between information quality

and expected returns (and volatility) varies endogenously in our model. As such, our model may help reconcile the apparently conflicting empirical evidence documented about this relation at the firm level (see Section 5.2).

The rest of the paper is organized as follows. We discuss the related literature in the next section. Section 3 presents the setup of the general model. In Section 4, we solve the static version of the model, which allows us to highlight the intuition for many of our results transparently. Section 5 analyzes the dynamic model in generality and presents our main results, and Section 6 concludes. Unless otherwise specified, all proofs are in the appendix.

2 Related Literature

While the majority of the RE literature has focused on linear-normal equilibria, a number of papers have explored the effects of relaxing the assumption that fundamental shocks and signals are normally distributed (e.g., Ausubel, 1990; Foster and Viswanathan, 1993; Rochet and Vila, 1994; DeMarzo and Skiadas, 1998; Barlevy and Veronesi, 2000; Spiegel and Subrahmanyam, 2000; Breon-Drish, 2010; and Albagli, Hellwig, and Tsyvinski, 2011). Our paper contributes to this literature by developing a model in the non-linearity arises endogenously. Specifically, even though shocks to fundamentals and signals are normally distributed in our model, since the uninformed investor is uncertain about whether other investors are informed, her beliefs about the price signal are given by a mixture of normals distribution.³

A similar non-linearity arises in the incomplete information, regime switching models of David (1997), Veronesi (1999), David and Veronesi (2008, 2009), and others, in which a representative investor updates her beliefs about which macroeconomic regime she is currently in using signals about fundamental shocks (e.g., dividends). In these models, the non-linearity in representative investor's filtering problem leads to time-variation in expected returns and stochastic volatility. Stochastic volatility also arises in noisy rational expectations models, like Fos and Collin-Dufresne (2012), in which noise trader volatility is stochastic and persistent. In contrast, these features arise *endogenously* in our model even though shocks to both fundamentals and news are i.i.d., and are driven by how uninformed investors learn to use the price to update their beliefs about fundamentals.

³In a series of papers, Easley, O'Hara and co-authors analyze the probability of informed trading (PIN) in a sequential trade model similar to Glosten and Milgrom (1985) (e.g., Easley, Kiefer, and O'Hara, 1997a; Easley, Kiefer, and O'Hara, 1997b; Easley, Hvidkjaer, and O'Hara, 2002). In these papers, the risk-neutral market maker updates her valuation of the asset based on whether a specific trade is informed or not, but does not face uncertainty about the likelihood of informed traders in the market. In contrast, the uninformed investor in our model must update their beliefs, not only about the value of the asset, but also about the probability of other investors being informed, which leads to non-linearity in prices.

Within the microstructure literature, our model is most closely related to [Li \(2011\)](#), which generalizes the continuous-time, Kyle-model of [Back \(1992\)](#) to allow for uncertainty about whether the strategic trader is informed or not. As in our model, the filtering problem is multi-dimensional and the resulting pricing rule is non-linear. Our analysis differs both in its motivation and execution. Whereas [Li \(2011\)](#) focuses on the market microstructure implications (e.g., market depth, insider’s profit), we focus primarily on the asset pricing implications. Importantly, since our model considers risk-averse investors, we are able to analyze the effect of the uninformed investor’s non-linear learning problem on risk-premia and expected returns.

Finally, our model contributes to the differences of opinion (DO) literature, which has been important in generating empirically observed features of price and volume dynamics (e.g., [Harrison and Kreps, 1978](#); [Harris and Raviv, 1993](#); [Kandel and Pearson, 1995](#); [Scheinkman and Xiong, 2003](#); [Banerjee and Kremer, 2010](#)). With the exception of [Banerjee, Kaniel, and Kremer \(2009\)](#), the DO models in the literature have largely ignored the role of learning from prices, since investors agree to disagree about fundamentals, and therefore find the information in the price irrelevant.⁴ In our model, investors may exhibit differences of opinion (since all potentially informed investors believe their signals are payoff relevant), but uninformed investors still condition on prices to update their beliefs about fundamentals. In this sense, our model bridges the gap between the RE and DO approaches.

3 Model Setup

In this section, we present the setup of the model and discuss some of our assumptions.

Payoffs. There are two assets: a risk-free asset and a risky asset. The gross risk-free rate is normalized to $R \equiv 1+r > 1$. The risky asset pays a stream of dividends $d_t \sim \mathcal{N}(\mu, \sigma^2)$, which are i.i.d., and investors observe the realization of d_t at date t . For notational convenience, we normalize μ to zero without loss of generality.⁵ The aggregate supply of the risky asset is constant and equal to Z . Denote the price of the risky asset at date t by P_t , and denote the dollar return on the risky asset by Q_{t+1} , where

$$Q_{t+1} = P_{t+1} + d_{t+1} - RP_t.$$

Preferences. Investor i has mean-variance preferences over next period’s wealth, and trades

⁴In the DO model of [Banerjee et al. \(2009\)](#), investors agree to disagree but use the price to update their beliefs about higher order expectations, which is useful for them to speculate against each other.

⁵In the numerical example of Section 5.4, we set μ to a non-zero value to ensure that prices are non-negative for a large region of the parameter region.

competitively i.e., is a price taker. In particular, she submits a limit order, $x_{i,t}$, such that

$$x_{i,t} = \arg \max_x \mathbb{E}_{i,t}[W_{i,t}R + xQ_{t+1}] - \frac{\alpha}{2} \text{var}_{i,t}[W_{i,t}R + xQ_{t+1}], \quad (1)$$

and where $\mathbb{E}_{i,t}[\cdot]$ and $\text{var}_{i,t}[\cdot]$ denote her conditional expectation and variance, respectively, given date her t information set, denoted by $\mathcal{F}_{i,t}$, $W_{i,t}$ denotes her wealth, α denotes her risk-aversion, and $x_{i,t}$ denotes her trading position in the risky asset at date t . This implies that her optimal demand for the risky asset is given by

$$x_{i,t} = \frac{\mathbb{E}_{i,t}[Q_{t+1}]}{\alpha \text{var}_{i,t}[Q_{t+1}]} = \frac{\mathbb{E}_{i,t}[P_{t+1} + d_{t+1}] - RP_t}{\text{var}_{i,t}[P_{t+1} + d_{t+1}]}. \quad (2)$$

The optimal demand is analogous to an investor's demand in an overlapping generations (OLG) model (e.g., [Spiegel, 1998](#); [Banerjee, 2011](#)). This structure facilitates tractability and retains the key feature of a dynamic framework — namely, an investor's optimal portfolio depends not only on her beliefs about the fundamental dividend, but also on her beliefs about future prices. Finally, the market clearing condition in the risky asset is given by

$$\sum_i x_{i,t} = Z. \quad (3)$$

An equilibrium consists of a price process, P_t , for the risky asset and investor demands, $x_{i,t}$, that satisfy the above optimality and market clearing conditions for all i, t (i.e., (2) and (3), respectively).

Information and Beliefs. There are two groups of investors: the uninformed (denoted by U) and the potentially informed (denoted by θ). Investors within each group are identical and behave competitively, so for ease of exposition, we will often refer to the representative investor for each group (i.e., investor U is the representative investor for all uninformed investors, and investor θ represents all potentially informed investors). The θ investors are either informed (i.e., $\theta = I$) or not informed (i.e., $\theta = NI$), where the prior probability of being informed is $\pi_0 \equiv \Pr(\theta = I)$. Specifically, at date t , investor θ receives a signal $S_{\theta,t}$ of the form:

$$S_{\theta,t} = \begin{cases} d_{t+1} + \varepsilon_t & \text{if } \theta = I \\ u_{t+1} + \varepsilon_t & \text{if } \theta = NI, \end{cases}$$

where $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ and $u_{t+1} \sim \mathcal{N}(0, \sigma^2)$ are independent of d_{t+1} .

It is convenient to parametrize the information quality of the informed investors' signal

(i.e., $S_{I,t}$) by the Kalman gain, λ , where

$$\lambda \equiv \frac{\text{cov}[S_{I,t}, d_{t+1}]}{\text{var}[S_{I,t}]} = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}.$$

Note that λ is decreasing in the noise of the signal (i.e., σ_ε^2) and takes values between zero and one. When $\lambda = 0$, $S_{I,t}$ is completely uninformative; investors learn nothing about future dividends by observing it. Conversely, when $\lambda = 1$, $S_{I,t}$ perfectly reveals the realization of next period's dividend. Unless otherwise noted, we assume $\lambda > 0$.

Investor U does not know θ and so is unsure whether the other investors have payoff relevant information about the asset. However, she has rational expectations about the joint distribution of signals and fundamentals. Since there are no additional sources of noise, one expects that in equilibrium, U will be able to infer $S_{\theta,t}$ from the price (i.e., P_t) and the aggregate residual supply (i.e., $Z - x_{\theta,t}$) and use this to update her beliefs about fundamentals.⁶ We say that an equilibrium is a *signal-revealing*, if the equilibrium price and allocations reveal $S_{\theta,t}$ (but not θ) to the U investor.⁷ In Section 4, we show that the unique equilibrium in the static model is signal-revealing. In Section 5, we characterize properties of signal-revealing equilibria in the general (dynamic) model. Moreover, we show that when the U investor faces no uncertainty about θ (i.e., $\pi_0 = 0$ or $\pi_0 = 1$), the unique stationary equilibrium of the dynamic model is, again, signal-revealing.

Let π_t denote the probability that investor U attributes to investor θ being informed at date t , i.e., $\pi_t \equiv \Pr_{U,t}(\theta = I)$. Then, investor U 's conditional beliefs about the value d_{t+1} next period are given by

$$\mathbb{E}_{U,t}[d_{t+1}] = \pi_t \lambda S_{\theta,t}, \text{ and} \quad (4)$$

$$\text{var}_{U,t}[d_{t+1}] = \pi_t \sigma^2 (1 - \lambda) + (1 - \pi_t) \sigma^2 + \pi_t (1 - \pi_t) (\lambda S_{\theta,t})^2. \quad (5)$$

Investor U will also update her beliefs about whether θ is informed conditional on the realization of d_{t+1} using Bayes rule. In particular, the posterior probability of $\theta = I$, conditional on a realization of d_{t+1} , is given by

$$\pi_{t+1} = \frac{\pi_t \Pr(S_{\theta,t} | \theta = I, d_{t+1})}{\pi_t \Pr(S_{\theta,t} | \theta = I, d_{t+1}) + (1 - \pi_t) \Pr(S_{\theta,t} | \theta = NI, d_{t+1})} = \frac{\frac{\pi_t}{\sigma_\varepsilon} \phi\left(\frac{S_{\theta,t} - d_{t+1}}{\sigma_\varepsilon}\right)}{\frac{\pi_t}{\sigma_\varepsilon} \phi\left(\frac{S_{\theta,t} - d_{t+1}}{\sigma_\varepsilon}\right) + \frac{1 - \pi_t}{\sqrt{\sigma^2 + \sigma_\varepsilon^2}} \phi\left(\frac{S_{\theta,t} - 0}{\sqrt{\sigma^2 + \sigma_\varepsilon^2}}\right)}, \quad (6)$$

⁶In our model, the equilibrium price alone does not reveal the signal $S_{\theta,t}$, since P_t is non-monotonic in $S_{\theta,t}$. We follow [Kreps \(1977\)](#) and allow the U investor to condition not only on equilibrium prices, but also equilibrium quantities.

⁷Formally, that $S_{\theta,t}$ is $\mathcal{F}_{U,t}$ -measurable.

where $\phi(\cdot)$ is the probability distribution function for a standard normal random variable.

For simplicity, we assume that investor θ believes that her signal is informative, and in particular, this implies that investor $\theta = NI$ has *incorrect* beliefs. This assumption is made for tractability since it implies that the conditional beliefs of the potentially informed investor are symmetric across types. For $\theta \in \{I, NI\}$, investor θ 's conditional beliefs about the value d_{t+1} next period are given by

$$\mathbb{E}_{\theta,t}[d_{t+1}] = \lambda S_{\theta,t}, \quad \text{and} \quad \text{var}_{\theta,t}[d_{t+1}] = \sigma^2(1 - \lambda). \quad (7)$$

Discussion of Assumptions. In our model, investors exhibit differences of opinion since $\theta = NI$ investors believe their signals are informative even though they are not. This lack of a common prior generates trade (e.g., [Milgrom and Stokey, 1982](#)). Alternatively, one could have assumed the existence of liquidity traders (e.g., $\theta = L$) who trade for non-informational reasons. While we expect that many of our results would be qualitatively similar in such a model, this alternative specification is significantly less tractable than our current setup.⁸ Moreover, given the empirical evidence on the trading behavior of retail investors, the assumption that some investors believe they are informed, even if they are not, seems quite plausible. As we discuss in [Section 5.2](#), the assumption also implies that our model generates novel predictions about the relation between information quality, investor disagreement and return characteristics.

As is common in the literature on asymmetric information in financial markets, we consider a single asset model. This assumption is made primarily for tractability. Given that there is empirical evidence for many of our predictions for individual stocks and for market returns (e.g., stochastic volatility, time-varying expected returns, volatility clustering), the mechanism that we highlight may be applicable at both the firm-level and the aggregate-level. For instance, one could interpret the risky asset in the model as an industry-level portfolio, which bears aggregate risk, and about which investors have asymmetric information. While a general equilibrium model with multiple firms or industries is beyond the scope of this paper, we expect the qualitative implications of the mechanism we highlight to survive in such a setting.

⁸In particular, the symmetry in the equilibrium trades of the I and NI investors in our specification is very useful for tractability. This implies that by just observing $x_{\theta,t}$, the U investor cannot update π — instead, she must observe d_{t+1} in order to update her beliefs about π . If the two types of θ traders had equilibrium trades from different distributions, this would lead the U investor to update π twice — once upon observing $x_{\theta,t}$, and then again, after observing d_{t+1} .

4 The Static Model

In this section, we present the static version of the model, i.e., when $Q_{t+1} \equiv d_{t+1} - RP_t$. The static version of the model allows us to solve for equilibrium prices in closed form and develop the underlying intuition for the model more transparently. In Subsection 4.1, we characterize how the equilibrium price changes as a function of the underlying parameters. In Subsection 4.2, we describe the model's predictions for observables like expected returns and volatility.

4.1 Equilibrium Prices

We begin with an explicit characterization of the price function. Note that since there is only one trading period, U investors do not trade subsequent to updating their beliefs about whether others are informed (i.e., after updating π_t). For notational consistency, we retain the t subscript, which can be interpreted as time zero. The uninformed investors prior, $\pi_t = \pi_0$, will be a key parameter of interest; comparative statics with respect to π_t in the static model will help develop the intuition for the dynamic model in which π_t evolves endogenously.

Proposition 1. *There exists a unique, signal-revealing equilibrium in the static model, in which the equilibrium price is given by*

$$P_t = \frac{1}{R} \left((\kappa_t + (1 - \kappa_t)\pi_t) \lambda S_{\theta,t} - \kappa_t \alpha \sigma^2 (1 - \lambda) Z \right), \quad (8)$$

where the weight κ_t is given by

$$\kappa_t = \frac{\sigma^2(1 - \pi_t\lambda) + \pi_t(1 - \pi_t)(\lambda S_{\theta,t})^2}{\sigma^2(1 - \lambda) + \sigma^2(1 - \pi_t\lambda) + \pi_t(1 - \pi_t)(\lambda S_{\theta,t})^2} \in [0, 1]. \quad (9)$$

Dollar returns (per share) are given by

$$Q_{t+1} \equiv d_{t+1} - RP_t = d_{t+1} - (\pi_t + (1 - \pi_t)\kappa_t) \lambda S_{\theta,t} + \kappa_t \alpha \sigma^2 (1 - \lambda) Z. \quad (10)$$

Note that the price can be decomposed into a market expectations component and a risk-premium component, since

$$P_t = \frac{1}{R} \left(\underbrace{\kappa_t \mathbb{E}_{\theta,t} [d_{t+1}] + (1 - \kappa_t) \mathbb{E}_{U,t} [d_{t+1}]}_{\text{expectations}} - \underbrace{\kappa_t \alpha \sigma^2 (1 - \lambda) Z}_{\text{risk premium}} \right). \quad (11)$$

Since the risk-aversion coefficient, α , and the aggregate supply of the asset, Z , scale the

risk-premium component, but not the expectations component. Thus, the product, αZ , determine the relative role of each component in the price. When risk aversion is low or the aggregate supply of the asset is small, the price is primarily driven by the expectations component. On the other hand, when risk aversion is high, or the aggregate supply of the asset is large, the risk-premium component drives the price. As such, it will be useful to characterize separately how each component of the price depends on the underlying parameters. The following corollary presents these results.

Corollary 1. *In the static model:*

- (i) *The expectations component of the price is increasing in $S_{\theta,t}$, and increasing in both λ and π_t for $S_{\theta,t} > 0$, but decreasing in λ and π_t for $S_{\theta,t} < 0$.*
- (ii) *The risk-premium component of the price is hump-shaped in $S_{\theta,t}$ around $S_{\theta,t} = 0$, U-shaped in π_t around $\frac{1}{2} \left(1 - \frac{\sigma^2}{\lambda S_{\theta,t}^2}\right)$, increasing in λ for small $|S_{\theta,t}|$, but decreasing in λ for large $|S_{\theta,t}|$ and $\lambda > 0$.*

The intuition for the expectations component is straightforward: the average conditional expectation about d_{t+1} is increasing in the signal $S_{\theta,t}$, the average expectation is more sensitive to the signal $S_{\theta,t}$ when either π_t or λ are larger, and so the derivative of this component with respect to π_t and λ , respectively, depend on the sign of $S_{\theta,t}$.

The risk-premium component of prices depends on the uncertainty that investors face. In particular, note that the risk-premium component can be rewritten as

$$-\kappa_t \alpha \sigma^2 (1 - \lambda) Z = -\alpha \left(\frac{1}{\text{var}_{\theta,t}[d_{t+1}]} + \frac{1}{\text{var}_{U,t}[d_{t+1}]} \right)^{-1} Z, \quad (12)$$

which is increasing in the conditional variance of both U and θ investors. Unlike standard RE models with linear equilibria, because the conditional variance of the uninformed investors depends on the signal realization, so too does the risk-premium component. Recall that the conditional variance $\text{var}_{U,t}[d_{t+1}]$ given in expression (5) is given by

$$\text{var}_{U,t}[d_{t+1}] = \pi_t \sigma^2 (1 - \lambda) + (1 - \pi_t) \sigma^2 + \pi_t (1 - \pi_t) (\lambda S_{\theta,t})^2. \quad (13)$$

Thus $\text{var}_{U,t}[d_{t+1}]$ increases in $|S_{\theta,t}|$, decreases in λ for small $|S_{\theta,t}|$, but increases in λ for sufficiently large $|S_{\theta,t}|$. Larger realizations of $|S_{\theta,t}|$ increase the uninformed investors' uncertainty about fundamentals, since they are unsure about whether the signal is informative. For small realizations of $|S_{\theta,t}|$, a more informative signal (i.e., high λ) reduces the posterior variance (for $\pi_t > 0$), but when $|S_{\theta,t}|$ is large enough, a more informative signal can increase the uninformed investor's uncertainty.

The overall effect of $S_{\theta,t}$ on the price in our model distinguishes it from linear rational expectations and difference of opinions models. Note that the expectations component of price is monotonic in $S_{\theta,t}$, but the risk-premium component is hump-shaped in $S_{\theta,t}$ around zero. This implies that the two components reinforce each other when $S_{\theta,t} < 0$, but offset each other when $S_{\theta,t} > 0$. In other words, the market reacts *asymmetrically* to news about fundamentals: the price responds more strongly to bad news (i.e., $S_{\theta,t} < 0$) than to good news (i.e., $S_{\theta,t} > 0$).

Since the risk-premium component is bounded by $-\alpha\sigma^2(1-\lambda)Z$, the expectations component dominates when $|S_{\theta,t}|$ is large enough. However, for $S_{\theta,t}$ small enough, the risk-premium component dominates. This means that for small, positive news (i.e., small $S_{\theta,t} > 0$), if the overall risk concerns in the market are large enough (i.e., αZ is large relative to $S_{\theta,t}$), the price actually *decreases* with $S_{\theta,t}$. Intuitively, a small, positive surprise about fundamentals can have a bigger impact on prices through the uncertainty it generates for uninformed investors than through its effect on the market’s expectations.

The mechanism through which the asymmetry in prices arises in our model differs from those in the regime-switching models of Veronesi (1999) and others. Specifically, in Veronesi (1999), the asymmetry in price reaction is driven by uncertainty about fundamentals, and so depends on whether the state of the economy is good or bad. The investor “over-reacts” to bad news only in good states, and “under-reacts” to good news only in bad states, because these are the instances when the signals increase uncertainty about the underlying state (and hence fundamentals). In our model, the asymmetry is not state-dependent: the price is more sensitive to bad news for *any* $\pi_t \in (0, 1)$ (not just when π_t is close to 1) and even when π_t is fixed. This is because the asymmetry is driven by uncertainty about the informativeness of the price signal.⁹

To summarize, the uncertainty that uninformed investors face about whether θ investors are informed leads to prices that react more strongly to bad news than to good news. Moreover, when the surprise is positive, but small, the price can decrease even when the news is good. These results are consistent with empirical evidence documented in the literature. For instance, using a sample of voluntary disclosures, Skinner (1994) documents that the price reaction to bad news is, on average, twice as large as that for good news.

4.2 Expected returns and volatility

Given the results from Proposition 1, we now turn to investigating the moments of returns.

⁹Note that in the dynamic version of our model, the uninformed investor updates π_t based on realizations of fundamentals (i.e., d_{t+1}), but this is not what drives the asymmetric reaction of P_t to $S_{\theta,t}$.

Proposition 2. *In the static model, the expected return and volatility, conditional on date t information (i.e., conditional on $S_{\theta,t}$), is given by*

$$\mathbb{E}_t[Q_{t+1}] = -(1 - \pi_t)\lambda\kappa_t S_{\theta,t} + \kappa_t\alpha\sigma^2(1 - \lambda)Z, \text{ and} \quad (14)$$

$$\text{var}_t[Q_{t+1}] = \sigma^2(1 - \pi_t\lambda) + \pi_t(1 - \pi_t)(\lambda S_{\theta,t})^2, \quad (15)$$

respectively. The unconditional expected return and volatility are given by

$$\mathbb{E}[Q_{t+1}] = \mathbb{E}[\kappa_t]\alpha\sigma^2(1 - \lambda)Z, \text{ and} \quad (16)$$

$$\text{var}[Q_{t+1}] = \sigma^2(1 - \pi_t^2\lambda) + (1 - \pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}] + (\sigma^2(1 - \lambda)\alpha Z)^2\text{var}[\kappa_t], \quad (17)$$

respectively.

To gain some intuition for the expressions in Proposition 2, note that the decomposition in (11) implies that dollar returns can be expressed as

$$Q_{t+1} = d_{t+1} - (\kappa_t\mathbb{E}_{\theta,t}[d_{t+1}] + (1 - \kappa_t)\mathbb{E}_{U,t}[d_{t+1}]) + \kappa_t\alpha\sigma^2(1 - \lambda)Z.$$

Since the U investor has rational expectations, we can condition down from her information set to characterize expected returns and volatility in returns from the perspective of the econometrician. Conditional on the econometrician's information set \mathcal{F}_t , the Law of Iterated Expectations implies that expected returns can be decomposed into an expectations component and a risk-premium component as follows:

$$\mathbb{E}[Q_{t+1}|\mathcal{F}_t] = \underbrace{\mathbb{E}[\kappa_t(\mathbb{E}_{U,t}[d_{t+1}] - \mathbb{E}_{\theta,t}[d_{t+1}])|\mathcal{F}_t]}_{\text{expectations}} + \underbrace{\mathbb{E}[\kappa_t\alpha\sigma^2(1 - \lambda)Z|\mathcal{F}_t]}_{\text{risk premium}}. \quad (18)$$

The expression for the conditional expected returns in Proposition 2 obtains from allowing the econometrician to condition on both $S_{\theta,t}$ and π_t , and the expression for unconditional expected returns assumes that the econometrician can only condition on π_t , but not the realization of $S_{\theta,t}$.¹⁰

Equation (18) highlights two important departures from standard linear models. First, since investor U 's conditional variance of d_{t+1} , and therefore κ_t , depends on the realization of $S_{\theta,t}$, the conditional risk-premium is state dependent (i.e., depends on $(\pi_t, S_{\theta,t})$). Second, unlike linear rational expectations models, the expectations component of expected returns need not always be zero in our model. In rational expectations models, since *every* investor's

¹⁰Formally, we assume that $S_{\theta,t}$ is measurable with respect to \mathcal{F}_{t+} , while π_t is measurable with respect to \mathcal{F}_t . Then the conditional moments are computed with respect to \mathcal{F}_{t+} while the unconditional moments are computed with respect to \mathcal{F}_t .

beliefs satisfies the Law of Iterated Expectations, and investors share a common prior, investors cannot disagree conditional on the same information set, and so the expectations component is zero. In our model, since investors exhibit differences of opinion, the expectations component of expected returns need not be zero provided $\pi_t < 1$. The uninformed θ investors (who believe they are informed) incorrectly condition on their signal, and this introduces a mean-reverting shock to expected returns. However, since the $S_{\theta,t}$ signals are mean-zero and i.i.d., the expectations component does not have an effect on the unconditional expected return.¹¹

Similarly, the Law of Total Variance implies that conditional on the econometrician's information set \mathcal{F}_t , the variance in returns can be expressed as the sum of the expected conditional variance and the variance of the conditional expected return i.e.,

$$\text{var}[Q_{t+1}|\mathcal{F}_t] = \mathbb{E}[\text{var}_{U,t}[d_{t+1}|\mathcal{F}_t] + \text{var}[\mathbb{E}_{U,t}[d_{t+1} - RP_t]|\mathcal{F}_t]. \quad (19)$$

In a static model, the only source of volatility in returns, conditional on date t information, is the dividend shock d_{t+1} . Hence, the conditional volatility is driven by the conditional variance of d_{t+1} , given the information at date t . The expression for the unconditional volatility of returns given in equation (17) can be decomposed into three terms, each of which captures a different source of risk,

$$\text{var}[Q_{t+1}] = \underbrace{\sigma^2(1 - \pi_t^2\lambda)}_{\text{fundamental}} + \underbrace{(1 - \pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}]}_{\text{expectations}} + \underbrace{(\sigma^2(1 - \lambda)\alpha Z)^2\text{var}[\kappa_t]}_{\text{risk premium}}. \quad (20)$$

The first term is the expectation of the conditional variance in returns and so captures the volatility in returns due to uncertainty about next period's fundamental dividend shock d_{t+1} . Note that this uncertainty disappears when both information is perfect (i.e., $\lambda = 1$) and it is commonly known that θ investors are informed (i.e., $\pi_t = 1$). The sum of the second and third terms are the variance of the conditional expectation of returns. The second term in (20) reflects the volatility in returns due to variation in the expectations component of conditional expected returns. Finally, the third term is volatility due to variation in the risk-premium component of conditional expected returns, and is zero when information quality is perfect (i.e., $\lambda = 1$). Much like prices, each of these components behaves differently with changes in π_t and other key parameters of interest, to which we now turn our focus.

¹¹Looking ahead, this will not be the case in the dynamic model, since the price will depend not only on expectations of future dividends, but also on expectations of future prices.

4.2.1 Comparative statics on return moments

To investigate comparative statics, we start by presenting the following result.

Proposition 3. *In the static model,*

- (i) *The unconditional expected return is homogeneous of degree 1 (HD1) in σ^2 and αZ .*
- (ii) *The unconditional volatility component due to fundamental shocks is HD1 in σ^2 and HD0 in αZ .*
- (iii) *The unconditional volatility component due to the expectations component of returns is HD1 in σ^2 and HD0 in αZ .*
- (iv) *The unconditional volatility component due to the risk premium component of returns is HD2 in σ^2 and αZ .*

As expected, (i) implies that unconditional expected returns are increasing in the fundamental volatility and the overall risk concerns in the market (as captured by αZ). Results (ii) through (iv) are also fairly intuitive, but they have important implications for which component drives overall volatility. In particular, when overall concerns about risk in the market are relatively high, the risk premium component of expression (20) is the key driver of overall return volatility. When αZ and σ^2 are relatively small, the first and second components of expression (20) drive overall volatility.

Proposition 3 is also useful for exploring comparative static results with respect to λ and π_t . For example, (i) implies that when exploring how expected returns change with λ and π_t , it is without loss to normalize σ^2 and αZ . By doing so, we are left with a two-dimensional parameter space (i.e., $(\pi_t, \lambda) \in [0, 1]^2$), over which the expected return can be plotted to obtain comparative-static results that obtain *for any parameter specification* of the model. Figure 1(a) illustrates the result. It implies that both higher quality information (as measured by λ) and greater likelihood of an informed trader (as measured by π_t) decrease the expected return. This is because both higher quality information and a higher likelihood of an informed trader imply that the price is more informative about the fundamentals, and the uncertainty faced by the uninformed investor is lower.

Using (ii) through (iv), we can conduct a similar exercise to to characterize the comparative static effects of each of the individual components of volatility. Figure 2(a) shows the volatility in returns due to fundamental dividend shocks is decreasing in π_t and λ , since an increase in either parameter reduces the uncertainty that investors face about next period's dividend. Figure 2(b) shows that the variance in the expectations component of conditional

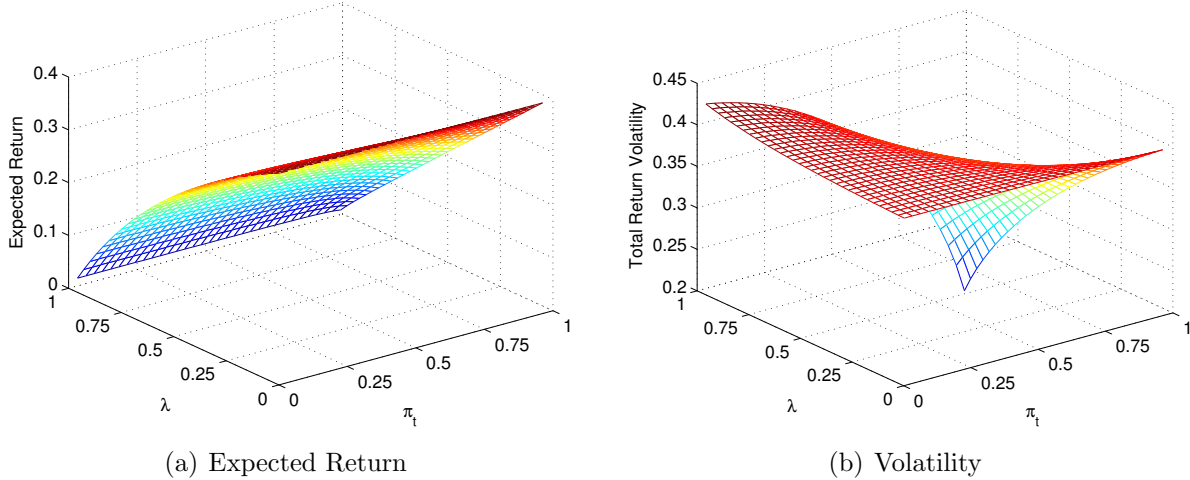


FIGURE 1: Illustration of expected returns (a) and total volatility (b) as they depend on the quality of information λ and the probability of a θ being informed, i.e., π_t . The other parameters are $\sigma = 40\%$, and $\alpha Z = 3$.

expected returns is decreasing in π_t but increasing in λ . Recall that the expectations component of the conditional expected returns is non-zero because investors exhibit differences of opinion, and in particular, because uninformed θ investors believe they are informed. This effect is larger when π_t is smaller (since θ investors are less likely to actually be informed) and when λ is larger (since uninformed θ investors put more weight on their signals), which leads to the effect on volatility. Figure 2(c) shows the risk-premium component of volatility is non-monotonic in both π_0 and λ . This is because the risk-premium component of returns is stochastic only when both λ and π_t are strictly between zero and one.

Of course, comparative statics on the total return volatility depend on the relative magnitudes of σ^2 and αZ , which determine the relative weight on each component. For instance, Figure 1(b) presents the effect of π_t and λ on overall volatility for a given set of parameters, for which the fundamental and expectations components dominate the risk-premium component.

5 The Dynamic Model

In this section, we extend our analysis to the dynamic setting. In addition to demonstrating that key features of the static setting are robust, the analysis offers several new comparative static results. It also allows us to generate testable predictions for both the time series of returns (Section 5.3) and the cross section of returns (Section 5.4).

The key insight to understanding the difference between the static and dynamic settings is that in the dynamic setting, the price is affected not only by investors' beliefs about fundamentals, but also their beliefs about future prices. This dynamic consideration reinforces

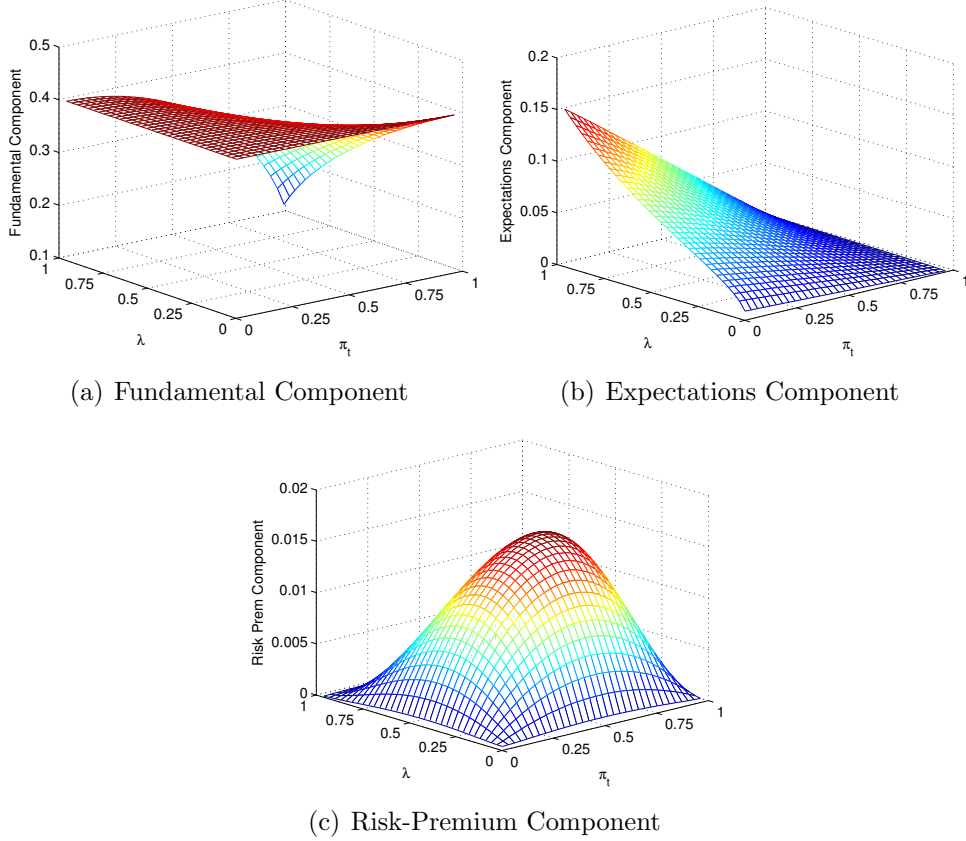


FIGURE 2: The three components of volatility as they depend on the quality of information λ and the probability of a θ being informed, i.e., π_t . Panel (a) plots the fundamental component of volatility (i.e., $\sigma^2(1 - \pi_t^2\lambda)$), panel (b) plots the expectations component (i.e., $(1 - \pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}]$), and panel (c) plots the risk-premium component (i.e., $(\sigma^2(1 - \lambda)\alpha Z)^2\text{var}[\kappa_t]$). The other parameters are set as in Figure 1.

some of the results from the static setting, but overturns others, and is crucial to understanding the intuition behind the model's predictions.¹² It is important to recognize that the degree to which these dynamic considerations and beliefs about future prices affect current prices depend on the extent to which investor discount future payoffs. In particular, as one would expect, return characteristics (i.e., expected returns, volatility) tend toward those in the static model as R increases. However, for our discussion and analysis, we assume a small enough value of R that allows us to highlight the differences in the implications of the dynamic and static models. We first provide a characterization of an equilibrium analogous to the one derived in the static setting.

Proposition 4. *In any signal-revealing equilibrium, investor i 's optimal demand is given by*

¹²As Banerjee (2011) argues, these dynamic considerations are also crucial in distinguishing standard rational expectations and difference of opinions models based on observable characteristics.

expression (2), investor beliefs are given by

$$\begin{aligned}\mathbb{E}_{U,t}[d_{t+1}] &= \pi_t \lambda S_{\theta,t}, & \mathbb{E}_{\theta,t}[d_{t+1}] &= \lambda S_{\theta,t}, \\ \text{var}_{U,t}[d_{t+1}] &= \sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)(\lambda S_{\theta,t})^2, & \text{and } \text{var}_{\theta,t}[d_{t+1}] &= \sigma^2(1 - \lambda)\end{aligned}$$

and the price of the risky asset is given by

$$P_t = \frac{1}{R} \left(\bar{\mathbb{E}}_t[P_{t+1} + d_{t+1}] - \alpha \kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z \right), \quad (21)$$

where $\bar{\mathbb{E}}_t[\cdot] \equiv \kappa_t \mathbb{E}_{\theta,t}[\cdot] + (1 - \kappa_t) \mathbb{E}_{U,t}[\cdot]$, and κ_t and λ are given by

$$\kappa_t = \frac{\text{var}_{U,t}[P_{t+1} + d_{t+1}]}{\text{var}_{U,t}[P_{t+1} + d_{t+1}] + \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]} \quad \text{and} \quad \lambda = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}. \quad (22)$$

The equilibrium price has an intuitive form; it is given by a weighted average of investors' conditional expectations about future payoffs, adjusted for a risk-premium. The weight of each investor's expectation in $\bar{\mathbb{E}}_t[\cdot]$ depends on the conditional variance of her beliefs relative to those of the others.¹³ The price reflects only the market expectation (i.e., $P_t = \frac{1}{R} \bar{\mathbb{E}}_t[P_{t+1} + d_{t+1}]$) if either investors are risk-neutral (i.e., $\alpha \rightarrow 0$), or, if the aggregate supply of the risky asset is zero (i.e., $Z = 0$).

As in the static case, the characterization of the price in Proposition 4 implies that (dollar) returns can be expressed as

$$Q_{t+1} = P_{t+1} + d_{t+1} - \bar{\mathbb{E}}_t[P_{t+1} + d_{t+1}] + \alpha \kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z.$$

And as before, since the U investor has rational expectations, we can condition down to the econometrician's information set \mathcal{F}_t using the Law of Iterated Expectations and the Law of Total Variance:¹⁴

$$\mathbb{E}[Q_{t+1}|\mathcal{F}_t] = \mathbb{E}[P_{t+1} + d_{t+1} - \bar{\mathbb{E}}_t[P_{t+1} + d_{t+1}] + \alpha \kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z|\mathcal{F}_t], \quad (23)$$

$$= \underbrace{\mathbb{E}[\kappa_t(\mathbb{E}_{U,t}[P_{t+1} + d_{t+1}] - \mathbb{E}_{\theta,t}[P_{t+1} + d_{t+1}])|\mathcal{F}_t]}_{\text{expectations}} + \underbrace{\mathbb{E}[\alpha \kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z|\mathcal{F}_t]}_{\text{risk premium}}, \quad (24)$$

$$\text{var}[Q_{t+1}|\mathcal{F}_t] = \mathbb{E}[\text{var}_{U,t}[P_{t+1} + d_{t+1}]|\mathcal{F}_t] + \text{var}[\mathbb{E}_{U,t}[P_{t+1} + d_{t+1} - RP_t]|\mathcal{F}_t]. \quad (25)$$

¹³Specifically, the weight on investor i is given by the precision of her conditional beliefs divided by the sum of the precisions of all investors, i.e., $\kappa_{i,t} = \frac{1/\text{var}_{i,t}[P_{t+1}+d_{t+1}]}{\sum_j 1/\text{var}_{j,t}[P_{t+1}+d_{t+1}]}$.

¹⁴As in the static case, we assume that $S_{\theta,t}$ is measurable with respect to \mathcal{F}_{t+} but not \mathcal{F}_t , while π_t is measurable with respect to \mathcal{F}_t . The conditional moments are computed with respect to \mathcal{F}_{t+} , and the unconditional moments are computed with respect to \mathcal{F}_t .

The decompositions in (24)-(25) will be useful for characterizing and understanding the properties of returns and volatility. As in the static case, the conditional risk-premium in our model is stochastic since investor U 's conditional variance of Q_{t+1} depends on the realization of $S_{\theta,t}$. Moreover, the expectations component of expected returns need not be zero when $\pi_t < 1$ since investors agree to disagree about in our model. The expectations component essentially reflects the component of expected returns that is driven by disagreement across investors.¹⁵ In contrast to the static version, however, the expectations component of the unconditional expected return need not be zero in the general case. As we discuss in Section 5.2, this is because the price is a non-linear function of $S_{\theta,t}$, and so while disagreement about next period's dividend may be unconditionally zero, the disagreement about next period's price is not.

5.1 Benchmark Cases

In this subsection, we consider two natural benchmarks which arise as special cases of the model. In both benchmarks, investor U is *not* uncertain about whether investor θ is informed. This exercise helps us to isolate this uncertainty as the key feature driving the main results in later sections. First, we characterize the equilibrium for the case in which $\pi_t = 0$. In this case, U investors do not condition on the price when updating their beliefs about the fundamental value of the asset and thus it is analogous to a Walrasian setting (or a standard difference of opinions setting). Next, we consider the other extreme, when $\pi_t = 1$. This setting is analogous to a standard rational expectations environment. In both of these special cases, and in contrast to the general model, the equilibrium price is linear in the signal $S_{\theta,t}$ of the θ investor.

Proposition 5. *Suppose $\pi_0 = 0$, and all θ investors are not informed. Then, there exists a unique, stationary equilibrium, which is signal-revealing, and where investor i 's optimal demand is given by expression (2), the price of the risky asset is given by*

$$P_t = A_0 S_{\theta,t} + B_0, \quad (26)$$

where $B_0 = -\frac{1}{r} \kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}] \alpha Z$ and A_0 is the unique real root to the following cubic equation:

$$R A_0 (2A_0^2 + (2 - \lambda) \lambda) - \lambda (A_0^2 + \lambda) = 0. \quad (27)$$

¹⁵ As discussed earlier, in a noisy rational expectations interpretation of the model, liquidity shocks or noise traders induce a similar component of expected returns.

Investor beliefs are given by

$$\begin{aligned}\mathbb{E}_{U,t}[P_{t+1} + d_{t+1}] &= B_0, \\ \mathbb{E}_{\theta,t}[P_{t+1} + d_{t+1}] &= B_0 + \lambda S_{\theta,t}, \\ \text{var}_{U,t}[P_{t+1} + d_{t+1}] &= A_0^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2, \text{ and} \\ \text{var}_{\theta,t}[P_{t+1} + d_{t+1}] &= A_0^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2(1 - \lambda),\end{aligned}$$

where λ and κ_t are given by equation (22). Since $\mathbb{E}_t[d_{t+1}|S_{\theta,t}] = 0$, conditional expected returns and variance in returns are given by

$$\mathbb{E}_t[Q_{t+1}|S_{\theta,t}] = \kappa_t \alpha \text{var}_t [Q_{t+1}|S_{\theta,t}] Z - RA_0 S_{\theta,t} \quad (28)$$

$$\text{var}_t[Q_{t+1}|S_{\theta,t}] = A_0^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2 = \text{var}_{U,t}[P_{t+1} + d_{t+1}] \quad (29)$$

Though θ investors are not informed, they believe they have payoff relevant information. As a result, the price responds to realizations of $S_{\theta,t}$. However, since the signals are spurious, prices are expected to mean-revert in the next period, and this induces mean reversion in expected returns through the $-RA_0 S_{\theta,t}$ term in expression (28). Finally, since the equilibrium price is linear in $S_{\theta,t}$, the risk-premium component of expected returns and the conditional volatility of returns are constant.

Proposition 6. *Suppose $\pi_0 = 1$ and all θ are informed. Then, there exists a unique, stationary equilibrium, which is signal-revealing, and where investor i 's optimal demand is given by expression (2), the price of the risky asset is given by*

$$P_t = A_1 S_{\theta,t} + B_1,$$

where $A_1 = \frac{\lambda}{R}$, and $B_1 = -\frac{1}{2r} \alpha \text{var}_{\theta,t}[P_{t+1} + d_{t+1}] Z$. Investor beliefs are given by

$$\begin{aligned}\mathbb{E}_{U,t}[P_{t+1} + d_{t+1}] &= \mathbb{E}_{\theta,t}[P_{t+1} + d_{t+1}] = B_1 + \lambda S_{\theta,t}, \text{ and} \\ \text{var}_{U,t}[P_{t+1} + d_{t+1}] &= \text{var}_{U,t}[P_{t+1} + d_{t+1}] = A_1^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2(1 - \lambda),\end{aligned}$$

where λ and κ_t are given by equation (22). Since $\mathbb{E}_t[d_{t+1}|S_{\theta,t}] = \lambda S_{\theta,t}$, conditional expected returns and variance in returns are given by

$$\mathbb{E}_t[Q_{t+1}|S_{\theta,t}] = \frac{1}{2} \alpha \text{var}_t [Q_{t+1}|S_{\theta,t}] Z \quad (30)$$

$$\text{var}_t[Q_{t+1}|S_{\theta,t}] = A_1^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2(1 - \lambda) = \text{var}_{U,t}[P_{t+1} + d_{t+1}] \quad (31)$$

When all θ investors are informed, prices respond efficiently to the information in $S_{\theta,t}$. As a result, expected returns are constant, and reflect only the risk-premium that investors require for holding the risky asset. The conditional volatility of returns is also constant, since the equilibrium price is linear in $S_{\theta,t}$.

5.2 Prices, Returns and Volatility

In the general case (i.e., $\pi_t \in (0, 1)$), the price is a non-linear function of $S_{\theta,t}$, d_{t+1} and π_t . As a result, the equilibrium price, which depends on conditional expectation and variance of next period's price for each investor, cannot be characterized analytically in closed form. Instead, we solve the general dynamic model numerically, by using an iterative procedure to compute the equilibrium price function (i.e., the fixed-point of (21)). Figure 3 illustrates the two components of the price. As in the static setting, the price P_t —the sum of the two components—is a non-linear function of π_t and $S_{\theta,t}$ and is more sensitive to bad news (i.e., negative $S_{\theta,t}$) than it is to good news (i.e., positive $S_{\theta,t}$). Moreover, the intuition is the same as before. All else equal, investors' expectation of dividends next period, and hence the expectations component of prices as in Figure 3(a), increases in $S_{\theta,t}$. However, a surprise in $S_{\theta,t}$ in either direction also leads to an increase in uncertainty for the U investor, and hence the risk-premium component is hump-shaped in $S_{\theta,t}$ as in Figure 3(b). For negative $S_{\theta,t}$ these two effects reinforce each other, while for positive $S_{\theta,t}$, the effects offset each other, and this leads to the asymmetric reaction of prices to $S_{\theta,t}$. The comparative statics with respect to π_t are familiar from the static case — the sensitivity of the expectations component to $S_{\theta,t}$ increases in π_t and the risk-premium component is U shaped in π_t for any $S_{\theta,t}$.

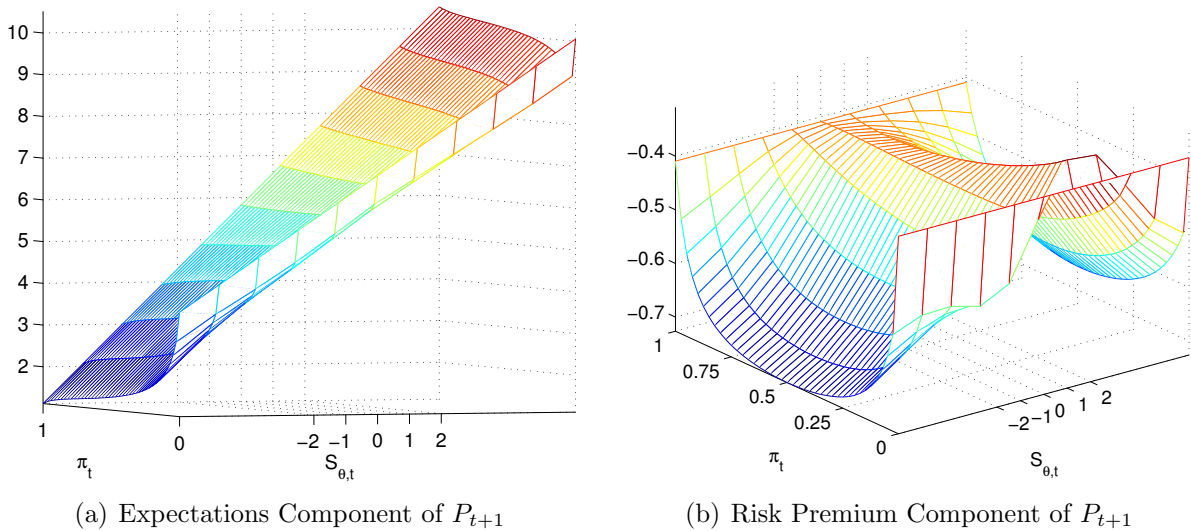


FIGURE 3: The two components of the equilibrium price function as they depend on the underlying state variable and the realization of information.

Unconditional expected returns can also be decomposed into an expectations component and a risk-premium component (see (24)). Unlike the result in Proposition 2, in a dynamic setting the expectations component of unconditional expected returns is not zero. As Figure 4(a) suggests, this is because the expectations component of expected returns depends not only on the difference in investors' unconditional expectations of d_{t+1} (which is zero, as in the static setting), but also the difference in their beliefs about P_{t+1} , which is not zero. The expectations component of expected returns given in (24) can be interpreted as a measure of disagreement between U and θ investors about future prices, it increases in λ and decreases in π_t . Intuitively, investors disagree more when λ is larger since each puts more weight on their own interpretation of $S_{\theta,t}$, but less when π_t is larger.

The risk-premium component of expected returns, shown in Figure 4(b), also behaves differently in the dynamic model. Recall that in the static setup, the risk-premium component decreases in λ and π_t since better information (i.e., higher λ) and a higher likelihood of θ being informed both lead to less uncertainty about next period's dividend. However, in the dynamic model, the risk-premium component is non-monotonic in both π_t and λ .

The difference in these patterns are a consequence of the fact that uncertainty about future prices affects the risk-premium component of expected returns. In particular, while an increase in λ decreases investor uncertainty about next period's dividends, it also makes prices more sensitive to $S_{\theta,t}$ thereby increasing future volatility. Similarly, for a fixed λ , the risk-premium component of the price is most sensitive to $S_{\theta,t}$ for intermediate values of π_t (see Figure 3(b)), and so uncertainty about future prices is higher for intermediate values of π_t . The interaction between dividend uncertainty and price uncertainty leads to the patterns in the risk-premium component. For high and low values of π_t , the effect of λ on dividend uncertainty dominates, and so the risk-premium decreases in λ . For intermediate values of π_t , the effect of λ on dividend uncertainty dominates for low λ but the effect on price uncertainty dominates for high λ , and hence the risk-premium exhibits a U -shape in λ . Finally, all else equal, uncertainty about next period's dividend (for the U investor) and about the risk-premium component of next period's price are highest for intermediate values of π_t , and hence the risk-premium exhibits a hump-shape in π_t .

As in the static case, the relative impact of the expectations component and the risk-premium component depend on fundamental volatility and overall risk concerns (i.e., αZ). Unlike the static setting, expected returns decrease in π_t when the expectations component dominates, but they are hump-shaped in π_t if the risk-premium component dominates. However, it is straightforward to show that expected returns and volatility are decreasing in λ when $\pi_t = 1$ but decreasing in λ when $\pi_t = 0$. The following proposition generalizes this

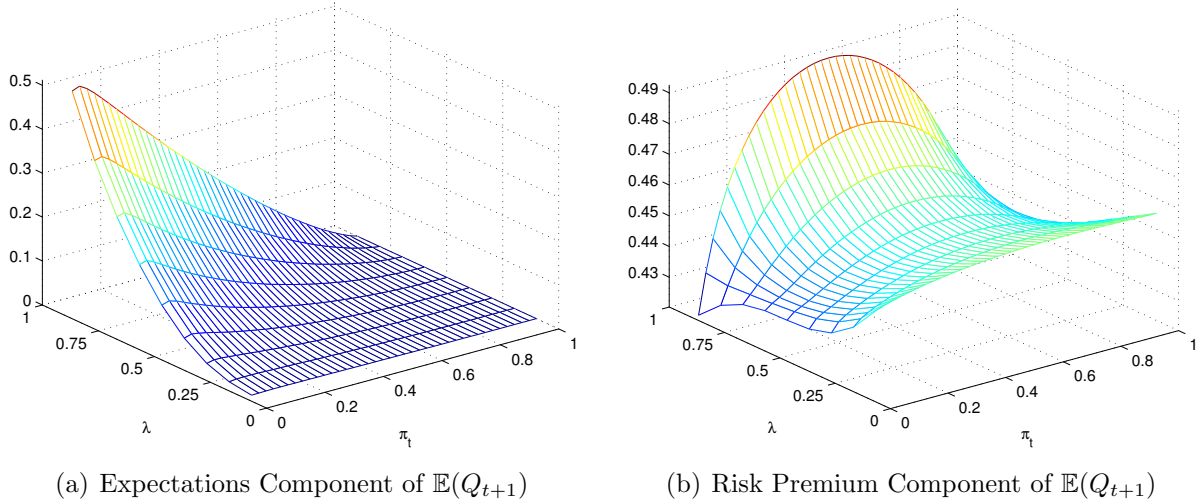


FIGURE 4: The two components of unconditional expected returns as they depend on the underlying state variable and the quality of the information.

result.

Proposition 7. *There exists a $\underline{\pi}, \bar{\pi} > 0$, $0 < \underline{\pi} < \bar{\pi} < 1$ such that:*

- *For all $\pi_t > \bar{\pi}$, expected returns are decreasing in λ .*
- *For all $\pi_t < \underline{\pi}$, expected returns are increasing in λ .*

As standard intuition suggests, when investors agree on the informativeness of $S_{\theta,t}$ (i.e., π_t is close to one), higher information quality (higher λ) leads to lower uncertainty and therefore lower expected returns. However, if investors disagree on the interpretation of the signal (i.e., π_t is close to zero), a signal with a higher λ generates more uncertainty for the U investor since it makes the θ investor trade more aggressively on his information. All else equal, this leads to higher volatility of current and future prices (due to higher sensitivity to $S_{\theta,t}$ shocks), which leads to higher expected returns.

Recall from equation (25) that using the law of total variance, we can decompose the total volatility (i.e., unconditional variance) of returns into into the expectation of the conditional variance, as in Figure 5(a), and the variance of the conditional expectation, plotted in Figure 5(b). Much like the analysis of expected returns, beliefs about future prices play an important role. First, note that the conditional variance in returns now depends not only on uncertainty about next period's dividend (as in the static setup) but also on uncertainty about next period's price. While uncertainty about next period's dividend is decreasing in both π_t and λ , uncertainty about next period's price can increase in λ (as P_{t+1} becomes more sensitive to $S_{\theta,t+1}$) and is hump-shaped in π_t , as discussed above. As a result, when dynamic considerations are important enough, the expected conditional variance component of the volatility in returns is hump-shaped in π_t and may increase or decrease in λ .

As in the static case, the variance of the conditional expectation of returns is increasing in λ — all else equal, a larger λ increases the sensitivity of investors' expectations to $S_{\theta,t}$ and, therefore, makes them more volatile. The effect of π_t on the variance of the conditional expectation of returns depends on which component dominates. In particular, as equation (24) suggests, the expectations component of conditional expected returns is driven by the disagreement between U and θ investors about expected future dividends and prices, and so the variance of this component decreases in π_t , since investors disagree less. However, the variance of the risk-premium component of expected returns is hump-shaped in π_t since the conditional risk-premium is most sensitive to $S_{\theta,t}$ when π_t is near $\frac{1}{2}$.

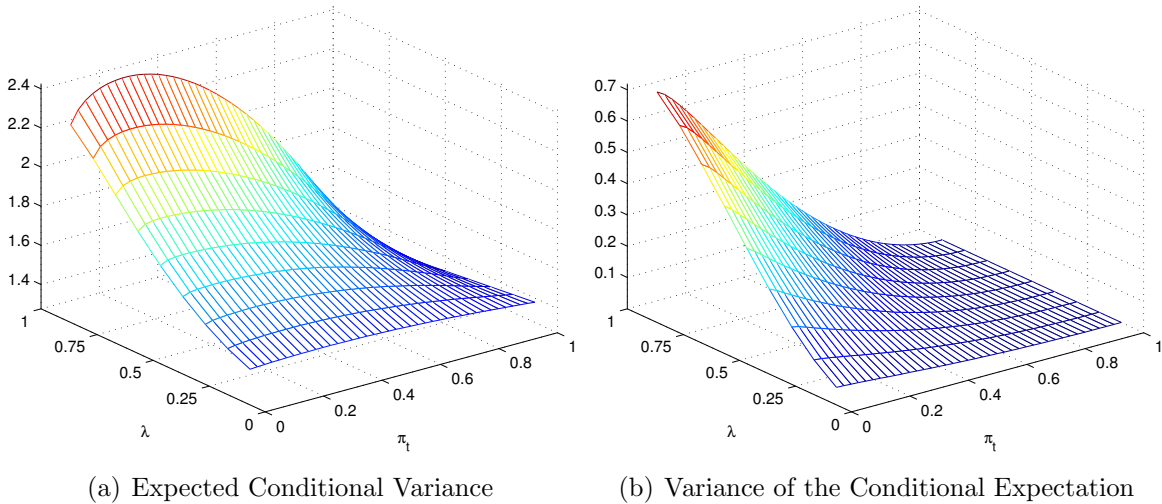


FIGURE 5: The two components of unconditional return volatility as they depend on the underlying state variable and the quality of the information.

The overall effect of π_t and λ on the variance of returns depend on the interaction of these components, which in turn, depend on the relative impact of the expectations and risk premium components of returns. Nevertheless, similar to expected returns, we have the following result.

Proposition 8. *There exists a $\underline{\pi}, \bar{\pi} > 0$, $0 < \underline{\pi} < \bar{\pi} < 1$ such that:*

- *For all $\pi_t > \bar{\pi}$, volatility is decreasing in λ .*
- *For all $\pi_t < \underline{\pi}$, volatility is increasing in λ .*

The empirical evidence documenting the firm-level relation between information quality and expected returns has been mixed. While some papers document a negative relation between information quality and expected returns (e.g., [Easley and O'Hara, 2005](#); [Francis, Nanda, and Olsson, 2008](#)), others find either limited or no evidence of a relation (e.g., [Core,](#)

Guay, and Verdi, 2008; Duarte and Young, 2009). Propositions 7 and 8 may be useful in reconciling the evidence documented in the literature, since they predict that the relation between information quality and expected returns / volatility depends on the likelihood that investors are informed (i.e., π_t), and as a result, can vary over time for an individual firm. The model suggests that conditioning on an empirical proxy of π_t (e.g., institutional ownership) may be useful in uncovering the underlying relation between information quality and expected returns and volatility. Finally, since the disagreement about dividend forecasts in our model decreases in π_t , our model suggests that the relation between proxies of disagreement (e.g., analyst forecast dispersion) and expected returns may also be time-varying.¹⁶

5.3 Time-Series Implications

The way in which λ and π_t affect the distribution of prices and returns allow us to generate predictions of the model that distinguish it from standard (linear) rational expectations models. The probability π_t that U investors attribute to θ investors being informed, is a stochastic (endogenous) state variable of the model and is persistent over time.¹⁷ As a result, in addition to generating *stochastic* expected returns and volatility, the model predicts that these conditional moments are *persistent*, despite the fact that shocks to fundamentals and signals are i.i.d.

The dynamic model generates predictability in future expected returns and volatility. In particular, if $\pi_t < 1$ is large enough, the model predicts *volatility clustering*— return surprises in either direction, driven by dividend surprises, are followed by an increase in volatility of returns and higher expected returns. Since Mandelbrot (1963), a large number of papers have documented the phenomenon of volatility clustering in various asset markets, and at different frequencies (see Bollerslev, Chou, and Kroner, 1992 for an early survey). The intuition for these results follows from the discussion in the earlier section. An unanticipated realization of d_{t+1} leads the U investor to revise her beliefs about θ being informed downwards (i.e., $\pi_{t+1} < \pi_t$).¹⁸ This revision in beliefs generates additional uncertainty for the U investors, and as a result, leads to higher future volatility and higher expected returns going forward (see Figures 4 and 5).

Figure 6 illustrates this clustering effect. Specifically, the figure plots expected returns

¹⁶The disagreement between the U and θ investors can be expressed as

$$|\mathbb{E}_{U,t}[d_{t+1}] - \mathbb{E}_{\theta,t}[d_{t+1}]| = (1 - \pi_t)\lambda|S_{\theta,t}|, \quad (32)$$

which is decreasing in π_t .

¹⁷From the perspective of the U investor it is a martingale; $\mathbb{E}_{U,t}[\pi_{t+1}] = \pi_t$.

¹⁸This follows from the evolution of π_t in (6) and because π_t is initially large.

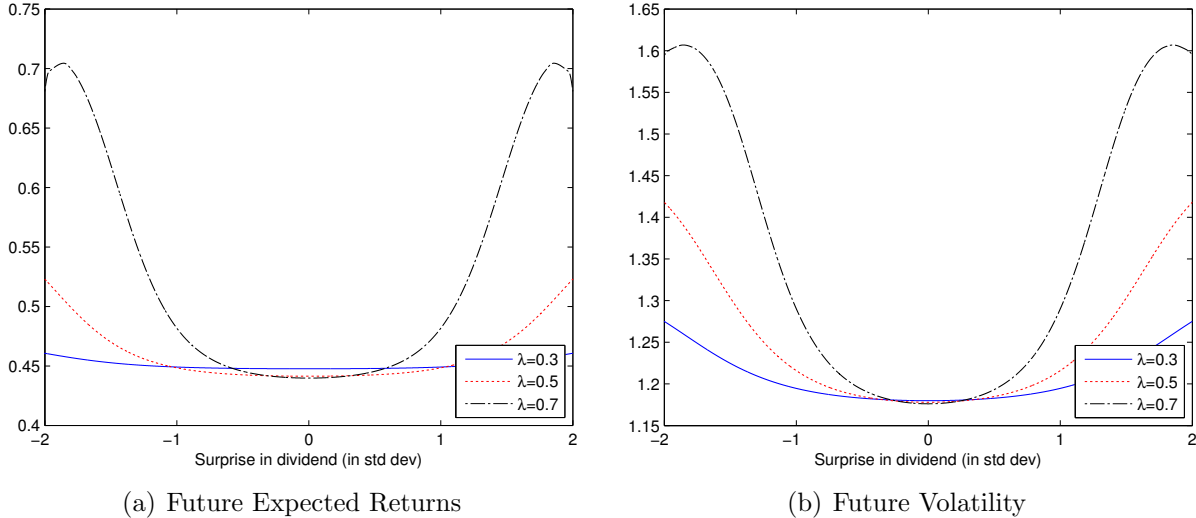


FIGURE 6: Expected returns and volatility in period $t + 1$ as they depend on dividend surprises in period t and the quality of the information. In both figures π_t is set equal to 0.9.

and volatility in period $t + 1$ as a function of the current realization of d_{t+1} (scaled by its standard error). Note that most realizations of $|d_{t+1} - S_{\theta,t}|$ lead to an increase in both volatility and expected returns. However, sufficiently extreme realizations of $|d_{t+1} - S_{\theta,t}|$, combined with sufficiently high λ , can actually decrease expected returns, since the posterior π_t is almost zero.

5.4 Numerical Example

In this section, we parametrize the model to gauge the magnitude and economic significance of our predictions. While the analysis in the earlier sections has focused on characterizing properties of dollar returns per share (i.e., Q_{t+1}), in this section, we characterize properties of the rate of return (i.e., $r_{t+1} = Q_{t+1}/P_t$) in order to highlight the robustness of the results and to facilitate comparisons to the broader literature.¹⁹ The risk-free rate is set to 5%, the mean of d_{t+1} is set to 3.5% and the supply of the asset is normalized to $Z = 1$.

The remaining two parameters are chosen so that, in the benchmark case where it is commonly known that I is informed (i.e., $\pi_t = 1$), the excess expected rate of return is 3% and the volatility in returns is 23%. In particular, we set $\alpha = 0.66$ and $\sigma = 1.18$. It is important to keep in mind that while dividends are extremely persistent in the data, they are assumed to be i.i.d. in our model. Hence, in order to match reasonable levels of expected returns and volatility, we must set the variance of dividends to be significantly higher than

¹⁹Because we are using normally distributed random variables, the population moments of r_{t+1} are not well defined due to prices arbitrarily close to zero (see [Campbell, Grossman, and Wang \(1993\)](#) and [Llorente, Michaely, Saar, and Wang, 2002](#) for a discussion). We adopt the conventional approach and choose μ large enough such that the numerical estimation of these moments is well behaved.

what is usually observed in the data.

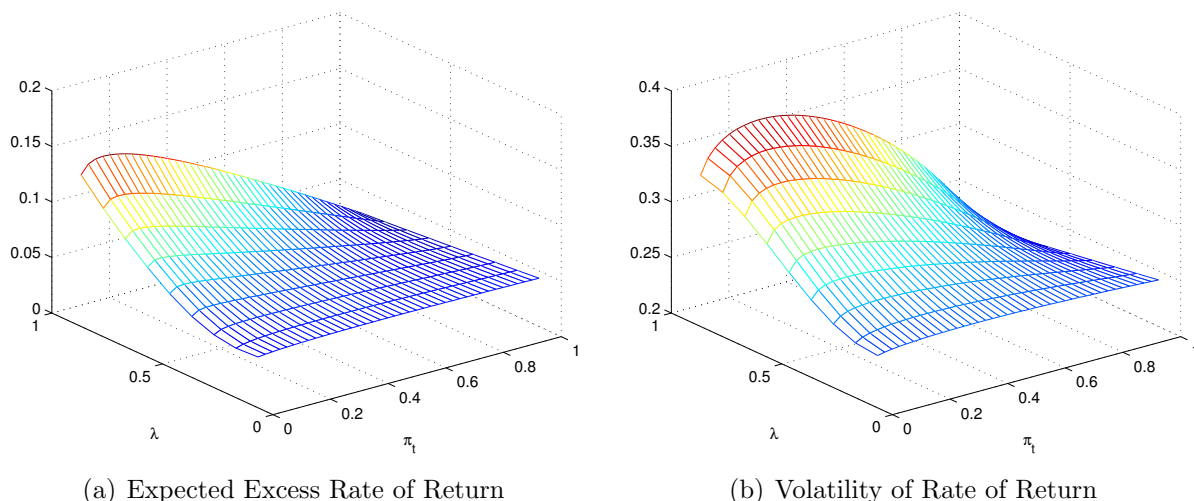


FIGURE 7: Illustrates the economic significance of the learning, after combining both the expectations component and the risk premium component, on the total of returns and volatility.

As Figure 7 shows, for this parametrization, both the expectations component and the risk premium component play a role; excess returns are first increasing in π_t (as in Figure 4(b)) but decreasing for larger π_t (as in Figure 4(a)) and thus expected returns are decreasing in π_t , except for π_t close to zero. Similarly, as expected excess returns and volatility are decreasing in λ for high π_t as better quality information reduces uncertainty when traders are sufficiently confident of its source. However, returns and volatility are *increasing* in λ for low π_t ; higher quality information leads to more volatile asset prices when traders are skeptical of the information source.

Figure 7 also gives a sense of the magnitude of the comparative statics results. For instance, an increase in λ from 0.3 to 0.7 implies an increase in expected returns from 4.6% to 6.9% and an increase in volatility from 26% to 31% percent (for $\pi_t = 0.5$); an increase in π_t from 0.3 to 0.7 implies a decrease in expected returns from 5.9% to 4.5% percent and a decrease in volatility from 29% to 26% percent (for $\lambda = 0.5$).

For the above parameters, Figure 8 provides magnitudes for the volatility clustering effect described in Section 5.3. For $\lambda = 0.7$, a one-standard deviation surprise in dividend realizations predicts an increase in future expected excess returns of roughly 1.5% (from 3.5% to 5%) and an increase in future volatility of 3.5% (from 23.3% to 26.8%). It is important to note that in the benchmark case where $\pi_t = 1$, these plots are perfectly flat. Thus, even for small deviations from the benchmark model ($\pi_t = 0.9$), the clustering effect can be quite economically significant.

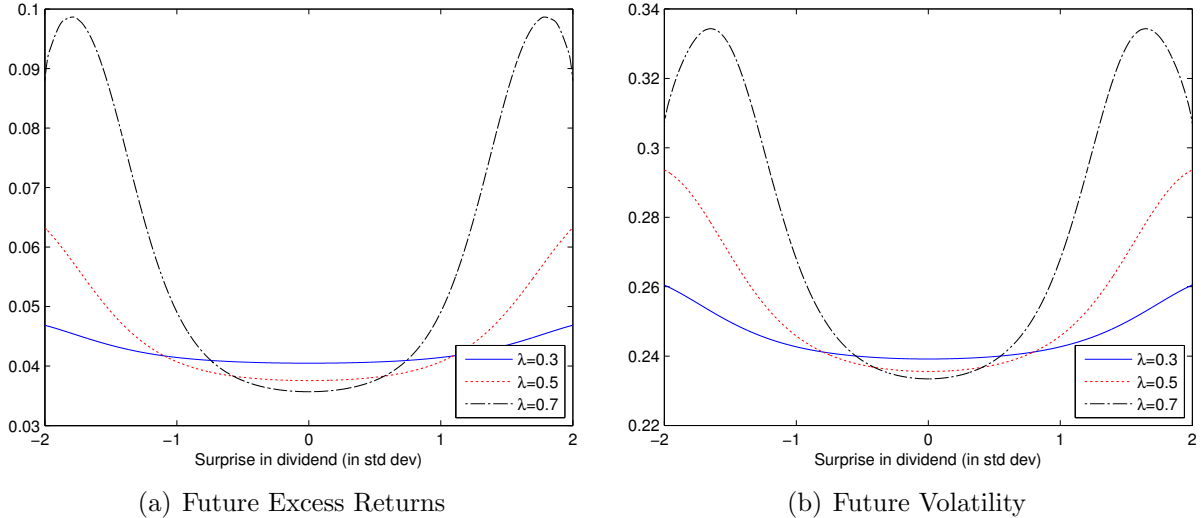


FIGURE 8: Illustrates the economic significance of the clustering effect. In both figures $\pi_t = 0.9$.

6 Final Remarks

This paper provides a framework for understanding how investors learn to use the information in prices to update their beliefs. We demonstrate that this type of learning has important implications for return dynamics. Specifically, the model generates non-linear prices, which are more sensitive to bad news than good news; persistent, stochastic expected returns and volatility, even though shocks to fundamentals and signals are i.i.d.; and volatility clustering (i.e., big return realizations of either sign are followed by higher volatility and higher expected returns).

We have kept the model as parsimonious as possible to highlight the key forces behind our results and to maintain tractability. For instance, the assumption that dividends are i.i.d. simplifies the numerical analysis significantly since it implies that the price function does not depend on the current level of dividends. But, while the model is able to generate non-trivial dynamics for returns and volatility, the lack of persistence in dividends limits the model's ability to empirically match moments of the data. Similarly, keeping the population of investors fixed simplifies the updating problem of the U investors, but implies that eventually U investors know whether θ investors are informed (i.e., π_t eventually converges to zero or one). Although beyond the scope of the current paper, we believe that an enriched version of the model, which incorporates persistence in fundamentals and allows θ to change over time, may be suitable for calibration. Our theoretical framework could also potentially be extended to study information acquisition in a dynamic setting, thereby endogenizing the information structure.

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Appendix

Proof of Proposition 1. If the equilibrium did not reveal $S_{\theta,t}$, then there would exist $S_{\theta,t}^1 > S_{\theta,t}^2$ for which P_t would be the same. But in this case, $x_{\theta,t}(S_{\theta,t}^1) > x_{\theta,t}(S_{\theta,t}^2)$, which implies that $S_{\theta,t}$ would be revealed in equilibrium. Moreover, since $\theta = I$ and $\theta = NI$ have symmetric optimal strategies, the equilibrium cannot reveal θ . Existence and uniqueness follow from characterizing the optimal demand for U and θ investors at date t in terms of their beliefs about next period's dividend d_{t+1} , and imposing the market clearing condition. \square

Proof of Corollary 1. Taking derivatives of κ_t we get

$$\frac{\partial}{\partial \lambda} \kappa_t = \frac{\sigma^2(1-\pi_t)(\pi_t S_{\theta,t}^2(2-\lambda)\lambda + \sigma^2)}{(\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2)^2} = \frac{(1-\pi_t)(\pi_t S_{\theta,t}^2(2-\lambda)\lambda + \sigma^2)}{\sigma^2(1-\lambda)^2} (1-\kappa_t)^2 \geq 0 \quad (33)$$

$$\frac{\partial}{\partial \pi_t} \kappa_t = -\frac{(\sigma^2 - (1-2\pi_t)\lambda S_{\theta,t}^2)\lambda(1-\lambda)\sigma^2}{(\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2)^2} = -\frac{(\sigma^2 - (1-2\pi_t)\lambda S_{\theta,t}^2)\lambda}{\sigma^2(1-\lambda)} (1-\kappa_t)^2 \quad (34)$$

$$\frac{\partial}{\partial S_{\theta,t}} \kappa_t = \frac{2\pi_t(1-\pi_t)(1-\lambda)\lambda^2\sigma^2 S_{\theta,t}}{(\sigma^2(1-\lambda) + \sigma^2(1-\pi_t\lambda) + \pi_t(1-\pi_t)(\lambda S_{\theta,t})^2)^2} = \frac{2\pi_t(1-\pi_t)\lambda^2 S_{\theta,t}}{\sigma^2(1-\lambda)} (1-\kappa_t)^2 \quad (35)$$

which implies κ_t is U -shaped around $S_{\theta,t} = 0$, and is hump shaped in π_t around $\frac{1}{2} \left(1 - \frac{\sigma^2}{\lambda S_{\theta,t}^2}\right)$.

- Effect of λ : The derivative of the expectations component of P_t with respect to λ is given by

$$\frac{\partial}{\partial \lambda} ((\kappa_t + (1-\kappa_t)\pi_t))\lambda S_{\theta,t} = (\pi_t + (1-\pi_t)(\kappa_t + \lambda \frac{\partial}{\partial \lambda} \kappa_t)) S_{\theta,t}$$

and so this component increases with λ for $S_{\theta,t} > 0$ and decreases in λ otherwise. The derivative of risk-premium component is given by

$$\frac{\partial}{\partial \lambda} (-\alpha\sigma^2(1-\lambda)\kappa_t Z) = -\alpha\sigma^2 Z ((1-\lambda)\frac{\partial}{\partial \lambda} \kappa_t - \kappa_t)$$

The derivative can be positive or negative depending on $|S_{\theta,t}|$. For $|S_{\theta,t}|$ small enough, the derivative is strictly positive, and so the risk-premium component of price increases in λ . But for $|S_{\theta,t}|$ large enough the derivative can be negative for intermediate λ , and so the risk-premium component of price is U -shaped in λ . Therefore, the risk premium is not monotone in λ . However, the risk-premium component is always lower for $\lambda = 0$ than it is for $\lambda = 1$.

- Effect of π_t : The derivative of the expectations component of P_t with respect to π_t is given by

$$\frac{\partial}{\partial \pi_t} ((\kappa_t + (1-\kappa_t)\pi_t))\lambda S_{\theta,t} = \left((1-\kappa_t) + (1-\pi_t)\frac{\partial}{\partial \pi_t} \kappa_t\right) \lambda S_{\theta,t}$$

Inserting the expression from (34) for $\frac{\partial}{\partial \pi_t} \kappa_t$ gives:

$$\left((1-\kappa_t) + (1-\pi_t)\frac{\partial}{\partial \pi_t} \kappa_t\right) = \frac{\sigma^2(1-\lambda) \left(((1-\pi_t)S_{\theta,t}\lambda)^2 + 2(1-\lambda)\sigma^2 \right)}{(1-\lambda)\sigma^2 + (1-\pi_t\lambda)\sigma^2 + (\lambda S_{\theta,t})^2(1-\pi_t)} > 0$$

Therefore, the derivative of the expectations component of prices with respect to π_t has the same sign as $S_{\theta,t}$.

The risk premium component of the price is U -shaped in π_t around $\frac{1}{2} \left(1 - \frac{\sigma^2}{\lambda S_{\theta,t}^2}\right)$, since

$$\frac{\partial}{\partial \pi_t} (-\alpha\sigma^2(1-\lambda)\kappa_t Z) = -\alpha\sigma^2(1-\lambda) Z \frac{\partial}{\partial \pi_t} \kappa_t$$

- Effect of $S_{\theta,t}$: The expectations component of P_t is always increasing in $S_{\theta,t}$, since

$$\frac{\partial}{\partial S_{\theta,t}} ((\kappa_t + (1 - \kappa_t)\pi_t))\lambda S_{\theta,t} = (\kappa_t + (1 - \kappa_t)\pi_t)\lambda + (1 - \pi_t)\lambda S_{\theta,t} \frac{\partial}{\partial S_{\theta,t}} \kappa_t > 0$$

The risk-premium component of the price is hump-shaped in $S_{\theta,t}$ around zero, since

$$\frac{\partial}{\partial S_{\theta,t}} (-\alpha\sigma^2(1 - \lambda)\kappa_t Z) = -\alpha\sigma^2(1 - \lambda)Z \frac{\partial}{\partial S_{\theta,t}} \kappa_t$$

This establishes the comparative static results. \square

Proof of Proposition 2. Taking the expectation of the right-hand side (RHS) of (10) and using that $\mathbb{E}_t[d_{t+1}] = \pi_0\lambda S_{\theta,t}$, we have

$$\mathbb{E}[Q_{t+1}] = \mathbb{E}[\mathbb{E}_t[Q_{t+1}]] = \alpha\sigma^2(1 - \lambda)Z\mathbb{E}[\kappa_t] - (1 - \pi_0)\lambda\mathbb{E}[\kappa_t S_{\theta,t}]$$

Thus, it suffices to show that $\mathbb{E}[\kappa_t S_{\theta,t}] = 0$. For this, note that $\kappa_t \cdot S_{\theta,t}$ is an odd-function (of $S_{\theta,t}$) and the distribution of $S_{\theta,t}$ is symmetric around zero. Thus, $\mathbb{E}[\kappa_t S_{\theta,t} | S_{\theta,t} > 0] = -\mathbb{E}[\kappa_t S_{\theta,t} | S_{\theta,t} < 0]$, which implies $\mathbb{E}[\kappa_t S_{\theta,t}] = 0$.

Volatility of returns is given by

$$\begin{aligned} \text{var}[Q_{t+1}] &= \mathbb{E}[\text{var}_t[Q_{t+1}]] + \text{var}[\mathbb{E}_t[Q_{t+1}]] \\ &= \mathbb{E}[\sigma^2(1 - \pi_t\lambda) + \pi_t(1 - \pi_t)(\lambda S_{\theta,t})^2] + \text{var}[\alpha\sigma^2(1 - \lambda)\kappa_t Z - (1 - \pi_t)\kappa_t\lambda S_{\theta,t}] \\ &= \sigma^2(1 - \pi_t^2\lambda) + (\alpha\sigma^2(1 - \lambda)Z)^2\text{var}[\kappa_t] + (1 - \pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}] \\ &\quad - 2\alpha\sigma^2(1 - \lambda)\lambda Z(1 - \pi_t)\text{cov}(\kappa_t, \kappa_t S_{\theta,t}) \end{aligned}$$

Recall that Stein's Lemma implies that for $Y \sim \mathcal{N}(0, \sigma_Y^2)$, and $g(Y)$ such that $\mathbb{E}[g(Y)Y] < \infty$ and $\sigma_Y^2\mathbb{E}[g'(Y)] < \infty$, we have $\text{cov}(g(Y), X) = \mathbb{E}[g'(Y)]\text{cov}(Y, X)$. This implies

$$\begin{aligned} \text{cov}(\kappa_t, S_{\theta,t}) &= \mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\text{var}(S_{\theta,t}) \\ \text{var}[\kappa_t S_{\theta,t}] &= \mathbb{E}[\kappa_t^2 S_{\theta,t}^2] - (\mathbb{E}[\kappa_t S_{\theta,t}])^2 \\ &= \text{cov}(\kappa_t^2 S_{\theta,t}, S_{\theta,t}) - \text{cov}(\kappa_t, S_{\theta,t}) \\ &= \left(\mathbb{E}\left[\kappa_t^2 + 2\kappa_t S_{\theta,t} \frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] - \mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\right)\text{var}(S_{\theta,t}) \\ \text{cov}(\kappa_t, \kappa_t S_{\theta,t}) &= \mathbb{E}[\kappa_t^2 S_{\theta,t}] - \mathbb{E}[\kappa_t]\mathbb{E}[\kappa_t S_{\theta,t}] \\ &= \text{cov}(\kappa_t^2, S_{\theta,t}) - \mathbb{E}[\kappa_t]\text{cov}(\kappa_t, S_{\theta,t}) \\ &= \left(\mathbb{E}\left[2\kappa_t \frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] - \mathbb{E}[\kappa_t]\mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\right)\text{var}(S_{\theta,t}) \end{aligned}$$

Since $\frac{\partial}{\partial S_{\theta,t}}\kappa_t(S_{\theta,t}) = -\frac{\partial}{\partial S_{\theta,t}}\kappa_t(-S_{\theta,t})$, we have that $\mathbb{E}\left[\frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] = 0$, and $\mathbb{E}\left[\kappa_t \frac{\partial}{\partial S_{\theta,t}}\kappa_t\right] = 0$. This implies that volatility can be expressed as:

$$\begin{aligned} \text{var}[Q_{t+1}] &= \sigma^2(1 - \pi_t^2\lambda) + (\alpha\sigma^2(1 - \lambda)Z)^2\text{var}[\kappa_t] + (1 - \pi_t)^2\lambda^2\text{var}[\kappa_t S_{\theta,t}] \\ &= \sigma^2(1 - \pi_t^2\lambda) + (\alpha\sigma^2(1 - \lambda)Z)^2\text{var}[\kappa_t] + \sigma^2(1 - \pi_t)^2\lambda \left(\mathbb{E}\left[\kappa_t^2 + 2\kappa_t S_{\theta,t} \frac{\partial}{\partial S_{\theta,t}}\kappa_t\right]\right) \end{aligned}$$

since $\lambda = \sigma^2/\text{var}(S_{\theta,t})$. \square

Proof of Proposition 3. It suffices to show that $\mathbb{E}[\kappa_t]$ and $\text{var}[\kappa_t]$ are HD0 in σ^2 , while $\text{var}[\kappa_t S_{\theta,t}]$ is HD1 in σ^2 . Recall that

$$\kappa_t = \frac{\sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 S_{\theta,t}^2}{\sigma^2(1 - \lambda) + \sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 S_{\theta,t}^2}$$

and by definition, $\lambda = \frac{\sigma^2}{\sigma^2 + \sigma_\epsilon^2}$ and $S_{\theta,t} \sim N(0, \sigma^2 + \sigma_\epsilon^2) = N(0, \sigma^2/\lambda)$, we have

$$\mathbb{E}[\kappa_t] = \frac{1}{\sqrt{2\pi\sigma^2/\lambda}} \int_{-\infty}^{\infty} \frac{\sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 s^2}{\sigma^2(1 - \lambda) + \sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 s^2} \exp\left(\frac{-s^2}{2\sigma^2/\lambda}\right) ds$$

Using a change of variables, by letting $x \equiv \frac{\sqrt{\lambda}}{\sigma} s$, we get that

$$\begin{aligned} E[\kappa_t] &= \frac{1}{\sqrt{2\pi\sigma^2/\lambda}} \frac{\sigma}{\sqrt{\lambda}} \int \frac{\sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 \frac{\sigma^2}{\lambda} x}{\sigma^2(1 - \lambda) + \sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 \frac{\sigma^2}{\lambda} x} \exp\left(\frac{-x^2}{2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda x}{(1 - \lambda) + (1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda x} \exp\left(\frac{-x^2}{2}\right) dx \end{aligned} \quad (36)$$

And clearly (36) is independent of σ . To see that $\text{var}[\kappa_t]$ is also independent of σ , note that same proof as above applies to $E[\kappa_t^2]$.

For $\text{var}[\kappa_t S_{\theta,t}]$, again using the same change of variables, we have that

$$\begin{aligned} \mathbb{E}[\kappa_t S_{\theta,t}] &= \frac{1}{\sqrt{2\pi\sigma^2/\lambda}} \int_{-\infty}^{\infty} \frac{\sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 s^2}{\sigma^2(1 - \lambda) + \sigma^2(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda^2 s^2} s \exp\left(\frac{-s^2}{2\sigma^2/\lambda}\right) ds \\ &= \frac{\sigma}{\sqrt{2\pi}} \int \frac{(1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda x}{(1 - \lambda) + (1 - \pi_t \lambda) + \pi_t(1 - \pi_t)\lambda x} \frac{x}{\sqrt{\lambda}} \exp\left(\frac{-x^2}{2}\right) dx \end{aligned}$$

which clearly scales with σ and hence $(E[\kappa_t S_{\theta,t}])^2$ scales with σ^2 . The same change of variables can be used to show that the same is true of $E[(\kappa_t S_{\theta,t})^2]$, which completes the proof. \square

Proof of Proposition 4. Optimality of $x_{i,t}$ follows from (2), the expressions for beliefs are given by (4)–(7), and the expression for the price follows from the market clearing condition, i.e., (3).²⁰ \square

Proof of Proposition 5. Given that dividends are i.i.d., and $S_{\theta,t}$ is uncorrelated with d_t , the price in any stationary equilibrium must be of the form $P_t = P(S_{\theta,t})$. This implies that $\mathbb{E}_{i,t}[P_{t+1}]$ and $\text{var}_{i,t}[P_{t+1}]$ are constant, and $\text{cov}[P_{t+1}, d_{t+1}] = 0$, which in turn implies that the optimal demand for each type of investor is linear in $S_{\theta,t}$ and P_t . Market clearing then implies that P_t is linear in $S_{\theta,t}$.

Given a conjecture for the price of the form $P_t = A_0 S_{\theta,t} + B_0$, the expressions for beliefs are immediate and they imply that the optimal demand for the U investor is given by

$$x_{U,t} = \frac{B_0 - R P_t}{\alpha (A_0^2 (\sigma^2 + \sigma_\epsilon^2) + \sigma^2)}.$$

²⁰To prove existence of such an equilibrium, it is sufficient to show that there exists a price function, $P(x)$, that satisfies the recursive equation given by (21) and where x denotes the current state, i.e., (π, S) . Formal details of the argument for existence of a fixed point are to be completed.

and for the θ investor is given by

$$x_{\theta,t} = \frac{B_0 + \lambda S_{\theta,t} - RP_t}{\alpha (A_0^2(\sigma^2 + \sigma_\varepsilon^2) + \sigma^2(1 - \lambda))}.$$

The market clearing condition implies that

$$RP_t = B_0 + \kappa_t \lambda S_{\theta,t} - \alpha \text{var}_{\theta,t}[P_{t+1} + d_{t+1}] \kappa_t Z$$

and matching terms with the conjectured price function, we get that A_0 solves

$$RA_0 = \frac{\lambda(A_0^2 + \lambda)}{2A_0^2 + (2 - \lambda)\lambda}$$

and $B_0 = -\frac{1}{r} \alpha \kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z$. This implies B solves the following cubic equation:

$$RA_0(2A_0^2 + (2 - \lambda)\lambda) - \lambda(A_0^2 + \lambda) = 0$$

Moreover, since the discriminant of the above equation is less than zero, there is one real root of A_0 . The above also implies that $0 \leq A_0 \leq \frac{\lambda}{R}$. The expressions for expected returns and volatility in returns follow from plugging in the expression for price and computing the moments. \square

Proof of Proposition 6. As in the proof of Proposition 5, the price in any stationary equilibrium must be a linear function of $S_{\theta,t}$. Given a conjecture for the price of the form $P_t = A_1 S_{\theta,t} + B_1$, the expressions for beliefs are immediate and they imply that the optimal demand for both types of investors is given by

$$x_{U,t} = x_{\theta,t} = \frac{B_1 + \lambda S_{\theta,t} - RP_t}{\alpha (\sigma^2(1 - \lambda) + A_1^2(\sigma^2 + \sigma_\varepsilon^2))}.$$

Since beliefs are identical, $\kappa_t = \frac{1}{2}$, and so the market clearing condition implies that

$$RP_t = B_1 + \lambda S_{\theta,t} - \frac{1}{2} \alpha \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z,$$

and matching terms with the conjectured price function, we get that $A_1 = \frac{\lambda}{R}$, $B_1 = -\frac{1}{r} \alpha \kappa_t \text{var}_{\theta,t}[P_{t+1} + d_{t+1}]Z$. The expressions for expected returns and volatility in returns follow from plugging in the expression for price and computing the moments. \square

Proof of Propositions 7 and 8. When $\pi_t = 1$, Corollary 6 implies that conditional expected return and variance in returns are constant, and so equal to the unconditional moments. Moreover,

$$\mathbb{E}[Q_{t+1}] = \frac{1}{2} \text{var}[Q_{t+1}] = \alpha Z \tag{37}$$

$$\text{var}[Q_{t+1}] = A_1^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2 (1 - \lambda) = \sigma^2 \left(1 - \lambda \left(1 - \frac{1}{R^2}\right)\right). \tag{38}$$

Differentiating w.r.t. λ , we get

$$\frac{\partial}{\partial \lambda} \text{var}[Q_{t+1}] = -\sigma^2 \left(1 - \frac{1}{R^2}\right) \tag{39}$$

which implies that expected returns and variance are decreasing in information quality.

When $\pi_t = 0$, Corollary 5 implies that

$$\kappa = \frac{A_0^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2}{A_0^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2 + A_0^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2 (1 - \lambda)} \quad (40)$$

$$= \frac{A_0^2 \sigma^2 / \lambda + \sigma^2}{A_0^2 \sigma^2 / \lambda + \sigma^2 + A_0^2 \sigma^2 / \lambda + \sigma^2 (1 - \lambda)} \quad (41)$$

$$= \frac{A_0^2 + \lambda}{2A_0^2 + 2\lambda - \lambda^2} = RA_0 / \lambda, \quad (42)$$

the unconditional expected return is

$$\mathbb{E}[Q_{t+1}] = \mathbb{E}[\kappa \alpha [A_0^2 (\sigma^2 + \sigma_\varepsilon^2) + \sigma^2] Z - RA_0 S_{\theta,t}] \quad (43)$$

$$= \sigma^2 RA_0 / \lambda (A_0^2 / \lambda + 1) \alpha Z \quad (44)$$

and the unconditional variance in returns is

$$\text{var}[Q_{t+1}] = \mathbb{E}[\text{var}_t[Q_{t+1}|S_{\theta,t}]] + \text{var}[\mathbb{E}_t[Q_{t+1}|S_{\theta,t}]] \quad (45)$$

$$= \sigma^2 ((1 + R^2) A_0^2 / \lambda + 1). \quad (46)$$

Since the unique solution to A_0 is given by

$$A_0 = \frac{\lambda}{6R} - \frac{12R^2\lambda - \lambda^2 - 6R^2\lambda^2}{3 \times 2^{2/3} R (72R^2\lambda^2 + 2\lambda^3 + 18R^2\lambda^3 + \sqrt{4(12R^2\lambda - \lambda^2 - 6R^2\lambda^2)^3 + (72R^2\lambda^2 + 2\lambda^3 + 18R^2\lambda^3)^2})^{1/3}} + \frac{(72R^2\lambda^2 + 2\lambda^3 + 18R^2\lambda^3 + \sqrt{4(12R^2\lambda - \lambda^2 - 6R^2\lambda^2)^3 + (72R^2\lambda^2 + 2\lambda^3 + 18R^2\lambda^3)^2})^{1/3}}{6 \times 2^{1/3} R}, \quad (47)$$

it can be shown that both the unconditional expected return and the unconditional variance in returns are increasing in λ .

Finally, since expected returns and volatility are continuous functions of π_t , the results follow. \square