Auctions of Real Options: Security Bids, Moral Hazard, and Strategic Timing

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Abstract

Assets with embedded real options are often sold to competing parties with bids partially contingent on the cash flows generated from exercising the options. Post-auction moral hazard arises because the winning bidder's real option differs from the auctioneer's. Moreover, as the valuations of an asset change over time, agents trade off auctioning immediately versus waiting for higher payoff. In formal auctions, security bids generally lead to significant investment delays and accelerations that are costly to the auctioneer's revenue and social welfare. In informal negotiations, post-auction investments are efficient and bidding equilibrium is equivalent to that of cash auctions. Strategic timing impacts bidding behavior, security ranking, equilibrium payoff, and investment efficiency in both formal and informal auctions. Optimal mechanisms entail weakly delayed auction and investment relative to efficient mechanisms, while bidders always initiate informal auctions. The paper thus offers the theoretical insight that auctions are not one-shot games, and cautions the use of security bids despite their oft-discussed benefits. By endogenizing the timing of auctions and characterizing the timing game, the paper also highlights the integral role of strategic timing in auction design. Finally when applied to the auctions of gas and oil leases, the model offers a potential explanation for the empirically documented large percentages of idle tracts of land.

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1 Introduction

On March 29, 2011, the U.S. Department of the Interior released a report showing that approximately 72% (26 million acres) of all offshore oil and gas leases and 57% (21 million acres) on federal lands remain idle, neither producing nor under active exploration and development by companies who hold those leases.[1] Ken Salazar, the Department Secretary, commented on the report, *"These are resources that belong to the American people, and they expect those supplies to be developed in a timely and responsible manner[...]we will also be exploring ways to provide incentives to companies to bring production online quickly and safely."*[2]

These statistics seem to illustrate the severe inefficiencies in the development of oil and gas fields that puzzle policymakers and researchers. In fact, oil and gas leases constitute a classic example of a broader category of projects and assets known as real options that often entail irreversible decisions in the face of uncertain investment opportunities. For example, commercial land developers time the market for construction of new houses; venture capitalists use a wide array of metrics to decide on whether and how much to continue funding a start-up; LBO funds may liquidate the firms after acquisition deals. The owners routinely sell the rights to control and exploit these real options through either formal auctions where the seller restricts the bids to a pre-specified ordered set and commits to allocate the asset by that order, or informal negotiations where bidders can make any bid and the seller chooses the most attractive one. The auction designs differ significantly: in addition to cash bids, security bids, whose values are contingent on future cash flows from the asset, have become increasingly common over the past few decades. For example, oil tract leases have been sold using cash contracts, bonus-bid contracts, royalty contracts and pure profit share contracts at various times.[3] This paper finds that moral hazard associated with various security bids is

[1] "Oil and Gas Lease Utilization - Onshore and Offshore. Report to the President" by U.S. Department of the Interior initially dated March 2011 and updated in May 2012.

[2] www.doi.gov, News Release of Office of the Secretary, U.S. Department of the Interior, March 29, 2011; http://washingtonindependent.com, the Washington Independent, March 30, 2011

[3] See Rothkopf and Engelbrecht-Wiggans (1992), Robinson (1984), Reece (1979), Hendricks, Porter, and Tan (1993), and Haile, Hendricks, and Porter (2010) for more details about the bidding for oil leases. In many auctions of large assets, aggressive bidders can declare bankruptcy post-auction, and the bids are essentially debt securities (Board (2007a)). A most notable case is the wireless spectrum auction for FCC C block bandwidth and the subsequent bankruptcy filing of NextWave Wireless Inc (Zheng (2001)). Equities, preferred convertibles, call options, and debts are frequently used in M&A and venture capital financing (Martin (1996),Povel and Singh (2010), Kaplan and Stromberg (2003) and Hellmann (2006)). Other examples include advance and royalty payments in publishing contracts (Dessauer (1981) and Caves

crucial to the understanding of inefficiencies in post-auction investments, and demonstrates that both the consideration of post-auction moral hazard and the strategic timing of auctions should play an indispensable role in designing these auctions.

Auctions of assets with embedded real options differ from traditional auctions in two important ways. First, the exercise of a real option often happens after the auction, which causes potential conflicts of interests between the auctioneer and the winning bidder. The intuition is simple: security bids can lead to contractual agreements that misalign the winning bidder's incentive with the seller's, and discourage the bidder's effort to optimally or efficiently exercise the investment option. As security bids become increasingly prevalent and the financial resources entailed are tremendous, understanding various post-auction moral hazard is important.[4] The second difference from traditional auctions is that the real option evolves over time, so do the private valuations of the bidders. The timing of auctions thus significantly influences security design, bidding behavior, and auctioneer's revenue and social welfare.[5] Studies illuminating the effects of strategic timing of auctions are thus important both for auction theory and for real life practices.[6]

This paper models the sale of a typical investment option, and explores the implications of post-auction moral hazard and strategic timing on dynamic investment and auction design. By doing so, it bridges the gap between auction theory and the literature on investment under uncertainty: the discussion of various selling mechanisms contributes to the emerging studies on contracting and agency issues under the real options framework; at the same time, the significant cost of post-auction moral hazard and its complicating effect on auction design are a timely cautionary tale to the use of security bids and conventional designs. To the author's knowledge, this paper is among the first to analyze post-auction moral hazard in auctions with contingent bids, and is the first to model the moral hazard of timing. It is also the first to cast strategic timing of auction as an optimal stopping problem for the

(2003)), motion picture deals (Chisholm (1997)), business licenses such as electronic gambling mchines with pre-specified profit tax, and military procurement contracts (DeMayo (1983) and McAfee and McMillan (1987a)).

[4]In the Gulf of Mexico alone, the oil and gas leases auctioned by the U.S. federal government from 1954 to 2007 have exceeded 300 billions and have used various forms of security bids.

[5]Though a popular topic for informal discussions among the eBay users, formal models of strategic timing of auctions have not been considered in the existing literature. Most other dynamic models of auctions examine multi-units sales or seller's commitment to no resale.

[6]Closed related is the timing of M&A in the real options framework because M&As can be viewed as informal auctions. For example, Gorbenko and Malenko (2013) examine takeover attempts in a real options framework with strategic timing by the bidders.

auctioneer, characterize the bidding equilibria and initiation strategies in informal auctions, and explore their implications on auction design and corporate investment. The domination of cash bids in large bidders markets and decreasing revenue and welfare with increased competition due to agency issues are novel in the literature as well.

The model concerns the moral hazard of investment timing and involves three sequential stages. In the first stage, an auctioneer wants to sell a project with an investment option. She decides when to sell an investment option, and designs the auction rules, potentially utilizing contingent securities. In the second stage, the bidders bid according to the auction rules and the winner gets allocated the project. In the final stage, the winning bidder rationally exercises the investment option to maximize her payoff, which involves a security payment to the auctioneer if the auction uses security bids. This paper focuses on the moral hazard of timing because investment timing is crucial in many projects of high option values, such as developments of real estate and natural resources, as well as transfer and licensing of technologies.[7] Capital allocation and investment timing are the two pillars underlying the theories of corporate investment. While moral hazard associated with the former has been well-studied, the one associated with the latter is equally important and deserves more attention.[8] Despite this, the intuition and many qualitative conclusions in this paper generalize to other types of moral hazard. Cong (2012a) discusses post-auction actions and moral hazard under more general settings.

If security bids lead to moral hazard, why use them in the first place? Prima facie, the type of security bids should not matter as there is always a cash equivalent bid. A key advantage to security-bid auctions that has received a lot of attention in the literature is that the use of contingent security effectively links price to a variable which is affiliated with bidders' private information (Milgrom (1985)). This is an example of the "linkage" principle that reduces the winning bidder's rent and enhances the seller's revenue. Contingent bids also mitigate liquidity constraints and reduce valuations gaps amongst various parties. As auctions with security bids typically involve post-auction actions, there is a tradeoff: contingent payments gives advantages such as the extraction of additionl surplus from the winning bidder, but they distort the effort for value creation.

This paper is related to the literature on security-bid auctions. Since Hansen's pioneering

[7]The transfer and licensing of technologies usually involve informal security-bids because nascent technology provides future development and growth options. For more details, see Choi (2001), Vishwasrao (2007), Datar, Frankel, and Wolfson (2001) and Bessy and Brousseau (1998).

[8]See Stein (2003) for some discussions on the former.

work (Hansen (1985)), many researchers have contributed to the discussion.[9] While the revenue-enhancing benefit of security bids has been extensively characterized (DeMarzo, Kremer, and Skrzypacz (2005)), Samuelson (1987) explains that adverse selection and moral hazard complicates the effect. Che and Kim (2010) demonstrate adverse selection effect could indeed outweigh the competitive benefits in security-bid auctions in terms of seller's expected revenue; Rhodes-Kropf and Viswanathan (2000) show adverse selection could lead to inefficiencies in security-bid auctions in bankruptcy reorganizations and privatizations. This paper examines post-auction moral hazard - the second issue Samuelson emphasized. Prior work typically ignores the inherent optionality, and as well as the dynamics of auctions and investments. The only other detailed study on this is Kogan and Morgan (2010) where the authors compare equity and debt auctions with moral hazard, both in theory and in a controlled experiment. Skrzypacz (2013) gives an overview. Also related are the theories of incentive contracting, typically applied to government procurement contracts.[10] This paper is unique in that it studies auctions with contingent bids in a dynamic setting, and analyzes the interaction of competition, moral hazard and auction timing.

This paper is also related to the recent literature on agency issues in the real options framework. Maeland (2002), Grenadier and Wang (2005), and Cong (2012b) study the distortion of investment incentives and timing due to adverse selection and moral hazard. Board (2007b) derives optimal selling mechanisms of options in discrete time while Maeland (2010) characterizes optimal screening contracts with first-hitting strategies. This paper differs foremost in considering the timing of auctions. It also derives optimal and efficient mechanisms in the entire strategy space under continuous-time setting and varying degrees of liquidity constraints.

The remainder of the paper is organized as follows. Section 2 introduces the real option and auction environment, and derives equilibrium bidding strategies in first-price auctions (FPAs) and second-price auctions (SPAs) with security bids. Section 3 exmamines post-auction investment and cost of moral hazard for some standard securities. Section 4 characterizes the optimal and efficient selling mechanisms, including the optimal timing. Section 5 analyzes informal auctions. Section 6 relaxes liquidity constraints and extends the model to allow combination of cash and security payment. Section 7 discusses model implications on

[9] Crémer (1987) and Riley (1988) are two early contributions.

[10] See Engelbrecht-Wiggans (1987), McAfee and McMillan (1986), Laffont and Tirole (1987) and McAfee and McMillan (1987b).

investment efficiency, auction design and auctioneer's strategic timing. Section 8 considers various extensions. Section 9 concludes. The appendix contains all the proofs.

2 Auctions with Security Bids

This section introduces the auction environment for a project with embedded real option, then establishes cash auctions as a benchmark, before finally analyzing FPAs and SPAs with security bids. Abandonment options can be similarly analyzed.

2.1 Auction Environment

An auctioneer owns an option to develop a project. The project requires a one-time irreversible development cost, and upon investment, generates a verifiable lump sum cash flow P_t, which evolves stochastically according to a geometric Brownian motion (GBM)

$$dP_t = \mu P_t dt + \sigma P_t dB_t, \tag{1}$$

where B_t is a standard Brownian Motion under the equivalent martingale measure, μ is the instantaneous conditional expected percentage change per unit time in P_t, and σ is the instantaneous conditional standard deviation per unit time.[11] The auctioneer either does not have the expertise or deems it too expensive to develop the project herself. There are N potential bidders who can develop the project at different private investment costs, with bidder i's cost being θ_i.[12] The seller only knows the prior distribution of the θ_is which is independent of P_t and has support $[\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} > 0$. The CDF and PDF are given by F and f respectively. All agents are risk neutral with discount rate r, where $r > \mu$.[13] Similar to DeMarzo, Kremer, and Skrzypacz (2005), the winning bidder has to pay an up-front cost $X > 0$ which we can interpret as the initial resources required by the project, or in the case of security issuance, the cost of underwriting, or simply an opportunity cost. To avoid

[11] The lump sum could represent the present value of a stream of expected future cashflows. The analysis and economic intuitions potentially extend to general linear diffussions and some jump processes, but GBM is standard in the real options literature for analytical tractibility.

[12] The analysis applies to cases where the bidders differ in other quantities, such as the capacities for production. Though not presented here, numeirical simulations suggest that the basic implications hold.

[13] $r > \mu$ is standard and ensures finite values. See McDonald and Siegel (1986) or Dixit and Pindyck (1994).

triviality, X is small enough so that the best type $\underline{\theta}$ is willing to participate in the auction if she owns the entire project.[14]

The bidders compete by offering security bids, which in real life are usually ranked by simple, easily implementable rules. As such, this section focuses on FPAs and SPAs of standard security bids that are well-ordered, notions formalized by a variant of the definitions in DeMarzo, Kremer, and Skrzypacz (2005):

DEFINITION A *standard security bid* is a promised payment made at the time of investment τ, and is represented by a function $S(P_\tau) \in [0, P_\tau]$ such that $S(P_\tau)$ and $P_\tau - S(P_\tau)$ are weakly increasing in P_τ. The function $S(s, P_\tau)$ for $s \in [s_L, s_H]$ defines an *ordered set of securities* such that (1) $S(s, \cdot)$ is a standard security; (2) $S(s, P_\tau)$ is continous and non-decreasing in s for all P_τ; (3) $S(s_L, P_\tau) \leq 0$ and $S(s_H, P_\tau) \geq P_\tau$.

$0 \leq S(P_\tau) \leq P_\tau$ corresponds to the limited liabilities for the seller and the bidder. To focus on contingent bids, this section assumes the seller and the potential bidders are liquidity-constrained that the project is the only pledgeable payment.[15] $S(P_\tau)$ and $P_\tau - S(P_\tau)$ need to be weakly increasing as is often required in the security design literature.[16] In addition to being standard, an ordered set of securities admit ranking with index s for any payoff from the project and the bids cover a range wide enough such that a participating bidder will always bid something and no bidder earns positive payoff at the highest allowed bid.

This section examines formal auctions in which the auctioneer restricts the bids to a well-ordered set of securities, and allocates the project to a bidder randomly chosen from the bidders with the same highest bid s. The winning bidder pays a security using the highest-bid index in FPAs or the next-highest-bid index in SPAs. This notion of an ordered set of securities is consistent with observations in real life: s could be the fraction of shares α in an equity auction $S(\alpha, P) = \alpha P$, the (negative) strike price k in a call option auction $S(-k, P) = \max\{P - k, 0\}$, or the conversion ratio c in a convertible debt auction $S(c, P) = \max\{cP, \min\{P, B\}\}$ for some promised face value B. Such securities are routinely used in

[14]In reality, X is often so small relative to the option value of the project that all bidders want to participate.

[15]DeMarzo, Kremer, and Skrzypacz (2005) show that seller's liquidity constraint necessarily follows from X not being verifiable. This rules out a solution as in Crémer (1987). Section 5 relaxes the assumption.

[16]Typical securities used in practice satisfy this assumption. See Hart and Moore (1995) and DeMarzo, Kremer, and Skrzypacz (2005) for more details.

M&As, VC contracts, and lease auctions where the winning bidder is indeed the one offering the highest s.

2.2 Cash Auctions as a Benchmark

This section discusses the bidding strategies in cash auctions and the post auction development of the project, as a benchmark for later sections. We work backwards and examine the post-auction investment of the project first. Upon winning, a bidder of type θ owns the project entirely, and optimally decides when to develop the project by paying the investment cost to maximize $\mathbb{E}[e^{-rt}(P_t - \theta)]$. This is a standard investment problem in the real options literature.[17] The optimal strategy involves a threshold cash flow level upon reaching which investment should take place immediately. Let P_0 denote the cash flow level when the auction is held. The value of the investment option W and optimal threshold $P^*(\theta)$ are independent of X and t, and are given by

$$W(P_0;\theta) = D(P_0; P^*(\theta))(P^*(\theta) - \theta), \tag{2}$$

$$P^*(\theta) = \max\left\{P_0, \frac{\beta}{\beta - 1}\theta\right\}, \tag{3}$$

where

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{4}$$

and

$$D(P; P') = \left(\frac{P}{P'}\right)^\beta, \quad P \leq P'. \tag{5}$$

As shown in Appendix A, $D(P_t; P')$ corresponds to the price at time t, of an Arrow-Debreu security that pays one dollar the first moment threshold $P' > P_t$ is reached. This interpretation allows an intuitive understanding of the option value of the project: it is the total value of Arrow-Debreu securities that replicate the payoff of the investment option at exercise. The option value can also be viewed as the value of the exercise payoff $P^*(\theta) - \theta$ discounted by the "expected discount factor" $D(P_0; P^*(\theta))$.

Bidder i's private valuation in the cash auctions is then $W(P_0; \theta_i) - X$, which is decreasing in θ_i. There exists a cutoff type for participation $\theta_c = \min\{\overline{\theta}, \theta_{BE}\}$, where the break-even

[17] For example, in McDonald and Siegel (1986) and Dixit and Pindyck (1994).

type θ_{BE} solves $W(P_0;\theta) - X = 0$ and is given explicitly as

$$\theta_{BE} = (\beta - 1)(P_0^{\beta}\beta^{-\beta}X^{-1})^{\frac{1}{\beta-1}}\mathbf{1}_{\{P_0>\beta X\}} + (P_0 - X)\mathbf{1}_{\{P_0\leq\beta X\}} \tag{6}$$

Types with costs higher than θ_c do not participate. From auction theory, FPAs and SPAs generate equivalent revenues $\mathbb{E}[\mathbf{1}_{\{\theta_{(2)}\leq\theta_c\}}(W(P_0;\theta_{(2)})-X)]$, and efficiently allocate the project to type $\theta_{(1)}$ if $\theta_{(1)} \leq \theta_c$, where $\theta_{(1)}$ and $\theta_{(2)}$ are the types with the lowest and second lowest investment costs in the auction. There is no post-auction moral hazard and the investments are efficient.

2.3 Equilibrium Bidding Strategies

Now for an auction with an ordered set of securities, the winning bidder of type θ rationally invests to maximize the present value of the project $U(t, s, \theta)$ where t is the time of investment and s is the index of the required security. Her private valuation is then

$$V(s, \theta) \equiv \max_{\tau} U(\tau, s, \theta) - X = \max_{\tau} \mathbb{E}_P[e^{-r\tau}(P_\tau - S(s, P_\tau) - \theta)] - X \tag{7}$$

Note that $V(s, \theta)$ is non-increasing in s because at the lower s, the winner can always pick the same stopping time as at the higher s, which yields a weakly higher value since $S(s, P)$ is weakly increasing in s. Similarly, $V(s, \theta)$ is increasing in θ.

In FPAs, the following regularity condition that most standard security-bid auctions satisfy is needed to ensure equilibrium bidding strategy and its proper characterization:

Condition (R1): $\ln V(s, \theta)$ is absolutely continuous in s with the derivative (when exists) being negative and decreasing in θ.

Proposition 2.1. *When (R1) holds, there exists a unique symmetric Bayesian Nash equilibrium for the first-price auction that is decreasing, differentiable, and is characterized by:*

$$s'(\theta) = \frac{(N-1)f(\theta)}{1-F(\theta)}\frac{V(s(\theta), \theta)}{V_1(s(\theta), \theta)} \tag{8}$$

for $\theta \leq \hat{\theta}$ with the boundary condition $V(s(\hat{\theta}), \hat{\theta})) = 0$, where the cutoff type $\hat{\theta} = \theta_c$. Types of $\theta > \hat{\theta}$ are not bidding.

All bidders participate if $\theta_c = \overline{\theta}$ as they can always bid security of zero value. If $\theta_c = \theta_{BE}$, cutoff type is bidding securities of zero value and breaks even in expectation. If she were bidding securities of positive value, either she fails to break even or the type slightly worse than her can come in and have a positive probability of reaping profit by bidding securities of zero value. This contradicts the very definition of a cutoff type. Thus the cutoff type is the same as in the cash-bid auction.

Next for a SPA, given the winning bidder of type θ invests rationally at τ, and let s' denote the highest bid among the $N-1$ remaining bidders, the net profit conditional on winning is

$$\mathbb{E}_{\theta_{-i}}[V(s',\theta)|s' \leq s] = \mathbb{E}_{\theta_{-i}}[\max_{\tau} \mathbb{E}_P[e^{-r\tau}(P_\tau - S(s',P_\tau) - \theta)]|s' \leq s] - X \tag{9}$$

The equilibrium bidding strategy is characterized by

Proposition 2.2. *The unique Bayesian Nash equilibrium in weakly undominated strategies in the second-price auction is for bidder i to increase the bid until $V(s,\theta) = 0$. The equilibrium strategy $s(\theta)$ is decreasing in θ and the cut-off type for participation is given by $\hat{\theta} = \theta_c$ with bid such that $V(s(\hat{\theta}),\hat{\theta}) = 0$. Types of $\theta > \hat{\theta}$ are not bidding.*

Again the cutoff type is the same as in the cash-bid auction and that type is bidding securities of zero value, for similar reasons as in FPAs. The next corollary follows directly from the fact that the bidding strategies are monotone.

Corollary 2.1. *In both FPAs satisfying (R1) and SPAs with security bids, the cutoff types are the same as in the cash auctions. Moreover, the investment option is allocated to the bidder with the least investment cost.*

In addition, the amount of competition as indicated by N, the initial commitment cost X, and the timing of the auction P_0 turn out to have fundamental impacts on the bidding behavior. The following proposition captures how varying these quantities would make bidders bid more or less aggressively.

Proposition 2.3. *The bidders bid more aggressively (weakly greater s for all types, and strictly greater s for a positive measure of types) in a FPA with security bids as N increases or X decreases, or if V/V_1 is increasing in P_0, as P_0 increases. They bid more aggressively in a SPA with security bids as X decreases, or if V is increasing in P_0, as P_0 increases .*

As a consequence, the security the winner pays has a greater index with more competition, smaller initial commitment costs, or (provided V/V_1 is increasing in P_0 if it is FPA, or V is increasing in P_0 if it is SPA) auctions held at higher cash flow levels. For example, in equity auctions discussed next, the shares the winning bidder pays is higher for bigger N, smaller X or greater P_0. Intuitively, a smaller X or higher P_0 correspond to higher valuation of the project to the bidders, which allows them to promise more to the auctioneer to increase their chances of winning. When N is bigger in a FPA, one has to increase the bid to outbid more competitors. However, this does not apply in SPA because one bidder's bidding strategy is independent of others' bids.

3 Moral Hazard in Post-auction Investments

This section applies the general bidding strategies derived earlier to a few specific security-bid auctions, in order to illustrate how post-auction investments can be delayed or accelerated based on the security design, and how these investment inefficiencies impact the auctioneer's revenue and social welfare. I work backwards to first derive the optimal strategies for each bidder post-auction conditional on rational participation and winning, and then solve for the equilibrium bidding strategies.

Equities/Royalties Bids and Investment Delays

Suppose α is the fraction of shares the winning bidder has to pay, i.e. $S(\alpha, P) = \alpha P$, her present value conditional on winning and exercising at τ is $\mathbb{E}_P[e^{-r\tau}[(1-\alpha)P_\tau - \theta] - X]$. The following proposition characterizes the optimal strategy for post-auction investment.

Proposition 3.1. *In auctions with equity bids, the winning bidder invests when the cash flow first reaches the threshold* $P^{equity}(\theta) = \max\{P_0, \frac{\beta\theta}{(\beta-1)(1-\alpha)}\}$.

The value to the winning bidder is then $V(\alpha, \theta) = D(P_0; P^{equity}(\theta))[(1-\alpha)P^{equity}(\theta) - \theta] - X$, which differs from that to the auctioneer $D(P_0; P^{equity}(\theta))P^{equity}(\theta)$. The incentives for investment are not aligned. In a FPA, since $\frac{\partial^2 \ln V(\alpha,\theta)}{\partial\alpha\partial\theta} < 0$ is well-defined except on the boundary $P_0 = \frac{\beta\theta}{(\beta-1)(1-\alpha)}$, (R1) is satisfied and $\alpha(\theta)$ is continuous and decreasing. The equilibrium bidding strategy is characterized by:

$$\alpha'(\theta) = \frac{(N-1)f(\theta)}{1-F(\theta)} \frac{V(\alpha(\theta), \theta)}{V_1(\alpha(\theta), \theta)} \tag{10}$$

where the boundary condition is $\alpha(\hat{\theta}) = 0$. In SPAs, the bidder θ increases α until $V(\alpha, \theta) = 0$. The investment threshold is $P^{equity}(\theta) = \max\{P_0, \frac{\beta}{\beta-1}\frac{\theta_{(1)}}{1-\alpha(\theta_{(2)})}\}$, and the cut-off type $\hat{\theta} = \theta_c$.

It is immediate that investment efficiency hinges on the cash flow level when the auction is held. For a winning bidder of type θ and a required share $\alpha > 0$, $P^{equity}(\theta) = P^*(\theta)$ if $P_0 > \frac{\beta\theta}{(\beta-1)(1-\alpha)}$. Otherwise, $P^{equity}(\theta) > P^*(\theta)$, and the investment is inefficiently delayed in general. When this happens, the investment timing is undesirable to a value-maximizing auctioneer because her value $D(P_0; P)S(\alpha, P)$ is also decreasing in P. Investing some time earlier could improve both the auctioneer's revenue and the welfare.

Call Options/Warrants Bids and Investment Accelerations

Let k be the strike price the winning bidder of type θ contracts, then $S(-k, P) = \max\{P - k, 0\}$. Her present value conditional on winning and exercising at τ is $\mathbb{E}_P[e^{-r\tau}[P_\tau - \max\{P_\tau - k, 0\} - \theta] - X]$. The following proposition characterizes the optimal strategy for post-auction investment.

Proposition 3.2. *In auctions with call option bids, a bidder of type θ always bids $k \in [X + \theta, P^*(\theta)]$, and upon winning, invests when the cash flow first reaches the threshold $P^{call}(\theta) = \max\{P_0, k\}$.*

Intuitively, a bidder does not bid too low a k that she would not break even upon winning, neither does she bid too high a k that the call option is never exercised since she can lower k to increase the chance of winning without reducing her payoff upon winning. Note $P^{call}(\theta) \leq P^*(\theta)$ with the equality holding when $P_0 > \frac{\beta}{\beta-1}\theta$. Inefficiency therefore lies in the potential *acceleration* of investments. Basically if the call option is going to be exercised, there is no incentive for the bidder to keep timing the market because that delays her payment k. When $P_0 \leq k$, $V = \frac{P_0^\beta(k-\theta)}{k^\beta} - X$, otherwise $V = k - \theta$.

Now for FPA, (R1) is satisfied and bidding equilibrium is described by Prop. 2.1.[18] Next for SPA, for those who participate, they bid up to their valuation, in other words, $k = \theta + X$ if $\theta < P_0 - X$ or k solves $\frac{P_0^\beta(k-\theta)}{k^\beta} = X$ otherwise. In either case, $k < \frac{\beta}{\beta-1}\theta$ and the investment

[18] $\frac{\partial \ln V}{\partial(-k)} < 0$. For $k \leq P_0$, $\frac{\partial^2 \ln V}{\partial(-k)\partial\theta} = -\frac{1}{(k-\theta-X)^2} < 0$; for $k > P_0$, it is $-\frac{(\beta-1)}{V^2 k^\beta}\left[\left(\frac{P_0}{k}\right)^\beta \frac{\theta}{\beta-1} - X\right] \leq -\frac{(\beta-1)}{V^2 k^\beta}\left[\left(\frac{P_0}{k}\right)^\beta [k-\theta] - X\right] = -\frac{(\beta-1)}{V k^\beta} < 0$ using the fact $k \leq \frac{\beta}{\beta-1}\theta$.

is strictly accelerated. In fact, the seller makes profit only when $\theta < P_0 - X$, otherwise the strike price is simply the value of the project, netting her zero profit. The cutoff type is the same as FPA.

Other standard securities can be analyzed in a similar fashion. The punchline is that post-auction investments could be inefficiently accelerated or delayed, and the timing is closely tied to the security being used in the auction. When the project has inherent option value to the bidders (P_0 being small), the accelerations and delays are more severe. The numerical illustrations to follow sometimes include another common form of security - the promise of payment B from the project's payoff. This is basically debts without interests, $S(B, P) = \min(P, B)$, also known as good loan, friendly debt, benevolent loan, or in Islamic finance, Qard/Qardul hassan. They are used frequently in contractual agreements in Islamic banking and microfinance, and are equivalent to granting the winning bidder instead of the seller call options-the exact opposite situation to that for call option bids. Compared to debt with interests, it allows the convenience to stay away from two-dimensional stopping problem, endogenous default consideartions, and liquidity constraints. It can be verified that investments are delayed using friendly debt.

Cost of the Post-auction Moral Hazard

To understand the cost of the post-auction moral hazard, let us examine how security bids enhance the seller's revenue in the first place. Suppose the winning bidder's θ is observable at the time of investment. The auctioneer can have a second-price profit auction, i.e., $S(\alpha, P) = \alpha(P - \theta)$. This linear contract, similar to those in the procurement literature leads to socially efficient investment, because the winning bidder faces a scaled optimal stopping problem and the incentives are fully aligned with the firm. If the cut-off type is bidding positive shares, then the type slightly worse than the cut-off type can participate and bid zero shares with a positive probability of winning, contradicting the former being the cut-off type of participation. Thus the cut-off types are the same for this auction and a cash-bid auction because the IR constraint they are subjected to are identical. Let $\theta_{(i)}$ denotes the ith smallest θ. If the second best type are not participating, the revenues in this SPA and in a cash auction are both zero. If the second best type is participating, i.e., $\theta_{(2)} < \hat{\theta}$ where $\hat{\theta}$ and bids up to its valuation with $\alpha(\theta_{(2)})$ shares that solves $(1 - \alpha)W(\theta_{(2)}) = X$.

With these, the seller's expected revenue in this SPA is

$$\begin{aligned}
&\mathbb{E}[\alpha(\theta_{(2)})W(\theta_{(1)})|\theta_{(2)} < \hat{\theta}] \\
=&\mathbb{E}[W(\theta_{(2)}) - (1-\alpha(\theta_{(2)}))W(\theta_{(2)}) + \alpha(\theta_{(2)})(W(\theta_{(1)}) - W(\theta_{(2)}))|\theta_{(2)} < \hat{\theta}] \\
=&\mathbb{E}[W(\theta_{(2)}) - X|\theta_{(2)} < \hat{\theta}] + \mathbb{E}[\alpha(\theta_{(2)})(W(\theta_{(1)}) - W(\theta_{(2)}))|\theta_{(2)} < \hat{\theta}] \qquad (11)
\end{aligned}$$

Note the first term is the seller's expected revenue in cash auctions and the second term the linkage benefit: for every realization of types such that the second lowest cost type participates, the seller recovers a portion $\alpha(\theta_{(2)})$ of the difference between the highest valuation and the second highest valuation, i.e., the winning bidder's information rent. This leads to a higher revenue than cash auctions, without harming social welfare.

In reality, θ is not observable, and as seen earlier, this leads to welfare losses due to inefficient investment timing. Table 1 Example 1 illustrates the impact on revenue and welfare of security bids. The profit auction yields higher revenue and the same welfare compared to the cash auctions, as predicted above. An equity auction also results in a higher revenue, but the welfare is reduced due to the inefficient investments.

Example 1: $P_0 = 15$, $N = 12$, $\theta \sim Unif[10, 50]$, $X = 2$, $\beta = 2.6$

Auction	Revenue	Welfare
Cash Auctions	0.7357	1.5957
Profit SPA	0.8962	1.5957
Equity SPA	0.7564	1.4559

Example 2: $P_0 = 15$, $N = 20$, $\theta \sim Unif[10, 50]$, $X = 0.5$, $\beta = 3$

Auction	Revenue	Welfare
Cash Auctions	2.3291	3.2509
Equity SPA	1.5274	1.7307
Call Option SPA	1.3778	2.9340

Table 1: Expected Revenues and Welfares

While some common security-bid auctions can still improve revenue, post-auction moral hazard is in general both costly to the revenue as well as to the welfare. Table 1 Example 2 gives an illustration: both equity SPA and call option SPA yield significantly lower revenues and welfares compared to the cash auctions. This inimical effect of post-auction moral hazard is similarly observed in FPAs too. Depending on the exact real option involved, one could lose more than 90% revenue and welfare using a security-bid auction compared to using a cash auction.

One way to get around this is to infer about the types from the bids and use that in the security design. For example, it can be verified that for FPA with bids on profit share, the bidding equilibrium exists and is characterized by Prop. 2.1. The seller can design an "inferred profit" auction defining profit as the difference of the payoff and the inferred type, and thus restore efficiency in equilibrium. However, such inferences would not eradicate the inefficiencies in call option auctions. Morever, though this resembles a FPA with profit share bids without post-auction moral hazard, and one might expect it to yield higher revenue than a cash auction due to the "linkage principle", the actual revenues of this "inferred profit" auction in the above two examples are exactly 0.7357 and 2.3291 - the same as in cash auctions. A natural question is whether one find security designs that ensure efficient investments and at the same time yield higher revenues than cash auctions. Section 6 explores these topics and shows that such an effort is futile: there is no security design that yields higher revenue than cash auctions while ensuring efficient investments.

4 Optimal and Efficient Mechanisms

This section derives the optimal (revenue-maximing) and efficient (welfare-maximizing) selling mechanisms of the investment project.[19] By the revelation principle, it suffices to examine a mechanism where all participating bidders report their types. Let $Q(\tilde{\theta}_i, \theta_{-i})$ be the probability of allocating the project to bidder i, where $\tilde{\theta}_i$ is the reported type by i and θ_{-i} are other participants' reported types. The contingent security then has the form $S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_\tau, \tau)$ at the time of investment τ where $\mathcal{I}_\tau$ is the set of contractible information including the historical time series of cash flow if that's contractible. The optimal investment time can be written generally as $\tau(\tilde{\theta}_i, S)$ where S is the contingent payment schedule. The expected utility to type θ_i upon participating is

$$U(\theta_i, \tilde{\theta}_i, \tau_i) = \mathbb{E}_{\theta_{-i}}\left[Q(\tilde{\theta}_i, \theta_{-i})\mathbb{E}_P\left[e^{-r\tau_i}(P_{\tau_i} - \theta_i - S(\tilde{\theta}_i, \theta_{-i}, \mathcal{I}_{\tau_i}, \tau_i)) - X\right]\right] \tag{12}$$

where $\mathbb{E}_P[e^{-r\tau_i}(P_{\tau_i} - \theta_i - S(\tilde{\theta}_i, \theta_{-i}, \{P_t, 0 \leq t \leq \tau_i\}, \tau_i))]$ is the post-auction utility.

[19] The analysis adopts the basic set up from earlier section, but can be extended to more general payoff functions of the real option and more general bidders' types, as long as the payoff from exercise $\pi(P,\theta)$ satisfies (1) $\frac{\partial \pi}{\partial P} > 0$, (2) $\frac{\partial \pi}{\partial \theta}$ is negative and continuous in θ, (3) the first-best optimal investment follows a threshold strategy that is increasing in θ, and finally (4) the "single-crossing" property $\frac{\beta}{P}\frac{\partial \pi}{\partial \theta} < \frac{\partial^2 \pi}{\partial P \partial \theta}$. This section corresponds to the simple case $\pi(P,\theta) = P - \theta$.

Incentive compatibility requires

$$U(\theta_i) \equiv U(\theta_i, \theta_i, \tau^*(\theta_i, S)) \geq U(\theta_i, \tilde{\theta}_i, \tau^*(\tilde{\theta}_i, S)) \tag{13}$$

and the individual rationality requires $U(\theta_i) \geq 0$.[20]

Lemma 4.1. *Any incentive compatible and individually rational mechanism satisfies*

$$U(\theta_i) = \mathbb{E}_{\theta_{-i}}\left[\int_{\theta_i}^{\overline{\theta}} Q(\theta, \theta_{-i})\mathbb{E}_P[e^{-r\tau_\theta}]d\theta\right] + U(\overline{\theta}) \tag{14}$$

where $U(\overline{\theta}) \geq 0$.

Proposition 4.1. *Assuming* $\theta + F(\theta)/f(\theta)$ *is increasing, an optimal mechanism exists and involves allocating the project to the agent with the lowest investment cost* θ *that is lower than a cutoff type* $\hat{\theta} = \min\{\overline{\theta}, \theta_{BE}^{optimal}\}$*, investing when* P_t *first reaches* $P_{\tau^*}(\theta) = \max\{P_0, \frac{\beta}{\beta-1}\left(\theta + \frac{F(\theta)}{f(\theta)}\right)\}$*, where* $\theta_{BE}^{optimal}$ *solves* $D(P_0; P_{\tau^*}(\hat{\theta}))[P_{\tau^*}(\hat{\theta}) - \hat{\theta}] = X$*. The contingent security payment schedule* S *satisfies:*

$$\begin{aligned}\mathbb{E}_{\theta_{-i}}\left[Q(\theta_i, \theta_{-i})S(\theta_i, \theta_{-i})\right] =& [1 - F(\theta_i)]^{N-1}[P_{\tau^*}(\theta_i) - \theta_i - X/D(P_0; P_{\tau^*}(\theta_i))] \\ & - \int_{\theta_i}^{\hat{\theta}} [1 - F(\theta)]^{N-1} D(P_{\tau^*}(\theta_i); P_{\tau^*}(\theta))d\theta.\end{aligned} \tag{15}$$

The assumption on $\theta + F(\theta)/f(\theta)$ makes the design problem regular and a sufficient condition satisfied by many common distributions is that the "inverse hazard function" $F(\theta)/f(\theta)$ being increasing in θ.[21] The allocation rule is the same as in the auctions with cash bids and security bids. However, it can be verified that the cut-off type is smaller, and the allocation outcome differs. This mechanism can be implemented by offering a FPA on investment cost: a menu of contracts with specific θ and the corresponding security $S(\theta, P)$ that pays

$$P_{\tau^*}(\theta) - \theta - \frac{X}{D(P_0; P_{\tau^*}(\theta))} - \int_{\theta}^{\hat{\theta}} \left(\frac{1 - F(\theta')}{1 - F(\theta)}\right)^{N-1} D(P_{\tau^*}(\theta); P_{\tau^*}(\theta'))d\theta' \tag{16}$$

[20] An extension with positive reserve utility or entry cost/fee is straightforward.

[21] Were θ a profit rather than cost, it would be exactly the inverse hazard rate. Regularity assumption is standard in the auctions literature, for example, see Krishna (2009).

when $P = P_{\tau^*}(\theta)$ and P otherwise. The project is awarded to the one bidding the lowest investment cost. Since the winning bidder only gets paid at $P_{\tau^*}(\theta)$, the moral hazard of investment timing is contracted away.

Unlike the mechanism derived in Board (2007b), the optimal mechanism with liquidity constraint involves security payment that is investment-contingent, a feature also in efficient mechanisms:

Proposition 4.2. *An efficient mechanism exists and involves allocating the project to the agent with the lowest investment cost θ that is lower than a cutoff type $\hat{\theta} = \theta_c$, investing when P_t first reaches $P_{\tau^*}(\theta) = \max\{P_0, \frac{\beta}{\beta-1}\theta\}$, and the contingent security payment schedule S satisfies (15) with the new $P_{\tau^*}(\theta)$.*

This cut-off is the same as in cash-bid auctions with endogenous entry. This mechanism is again implemented by a FPA on investment cost: a bid of θ has a corresponding security $S(P_{\tau^*}(\theta), P)$ that satisfies equation (16) when $P = P_{\tau^*}(\theta)$ and P otherwise.

The allocation rule is the same for both the optimal mechanism and the efficient mechanism, but the cut-off type is smaller in the optimal mechanism, some projects of positive social value are not invested. Moreover, the investment trigger under the optimal mechanism (Prop 4.1) is weakly higher than the social optimal $\frac{\beta}{\beta-1}\theta$ (Prop 4.2). But unlike Corollary 1 in Board (2007b), if $P_0 \geq \frac{\beta}{\beta-1}\left(\theta + \frac{F(\theta)}{f(\theta)}\right)$, the project is not invested later. These are summarized as

Corollary 4.2. *Suppose $F(\theta)/f(\theta)$ is non-decreasing, the optimal mechanism leads to investments being weakly delayed or missed entirely relative to the socially optimal outcome.*

In addition to designing the auction rules, the auctioneer has the liberty to decide when to hold the auction. Though the exact solution to the strategic timing problem depends on knowledge about the distribution of types, there are some general properties of the strategic timing in optimal and efficient mechanisms.

Proposition 4.3. *In optimal and efficient mechanisms, the auctioneer never sells the project at a cash flow level where she expects no potential bidder who would invest right away.*

If this were not the case, suppose she holds the auction at $P_{00} < \frac{\beta}{\beta-1}\underline{\theta}$. She would find it profitable to wait till a slightly higher cash flow level P_0 is reached because this does not affect the payoff from the real option in expectation, delays the incidence of auction friction

X and increases the probability of the project being allocated by increasing the cut-off type for participation.

Proposition 4.4. *A threshold strategy for timing the auction exists in both optimal and efficient mechanisms, and the threshold cash flow level solves*

$$\int_{\underline{\theta}}^{\hat{\theta}} Nf(\theta)[1 - F(\theta)]^{N-1}[\beta X - (\beta - 1)P^*(z(\theta)) + \beta z(\theta)]d\theta = 0 \tag{17}$$

where $z(\theta) = \theta + \frac{F(\theta)}{f(\theta)}$ *in optimal mechanisms and* $z(\theta) = \theta$ *in efficient mechanisms.*

In general the timing strategy could involve multiple thresholds. But in real life, X is usually much smaller than the value of the project that all types are participating. This leads to a unique threshold strategy for timing the auction that admits finer characterization:

Proposition 4.5. *When* $X < W\left(\frac{\beta}{\beta-1}\underline{\theta}, \overline{\theta} + 1/f(\overline{\theta})\right)$*, there is a unique threshold strategy for timing in both optimal and efficient mechanisms. The optimal mechanism design involves the auction being held weakly earlier than the socially efficient timing.*

[More discussion to be added]

While optimal and effience mechanisms assume the seller has full control to design the selling mechanism, but in real life bidders could play a more active role as discussed next.

5 Informal Auctions

So far the bidders are restricted to bid with contingent payments in a pre-specified ordered set in formal auctions. Such restrictions are infrequent in that the auctioneer often cannot ignore offers from outside the set, especially in informal negotiations. The auctioneer thus considers all bids and chooses the most desirable ex post. This essentially leaves the "security design" of the auction to the bidders as they can bid any contingent payments. It turns out investments are always efficient under such settings, and because of that, the conclusion in DeMarzo, Kremer, and Skrzypacz (2005) still applies: cash is the cheapest way for a better type to separate from lower types and in equilibrium everyone is using bids that are equivalent to cash bids.

5.1 The Signaling Game

As the auctioneer commits to neither a bidding rule nor an allocation rule, she chooses the bid that gives her the highest expected payoff based on her beliefs regarding the type of each bidder. The auction therefore exhibits features of a signaling game of the following form:

1. An auction is initiated.

2. Participating bidders submit informal bids simultaneously. An informal bid by bidder i is a payment $S^i(\cdot)$ contingent on actual payoff of the project such that only cash flow from the project is plegeable, i.e., $S^i(P) \in [0, P]$.

3. The auctioneer chooses the winning bidder rationally according to the valuation function $R(S^i) = \mathbb{E}[R_\theta(S^i)|\Theta(S^i)]$. $\Theta(S^i)$ is her belief of bidder i's type upon seeing the bid, and $R_\theta(S^i) = \mathbb{E}[e^{-r\tau_\theta^i} S^i(P_{\tau_\theta^i})]$ where τ_θ^i is the optimal stopping rule for type θ when bidding S^i, i.e. $\tau_\theta^i = \text{argmax}_\tau \, \mathbb{E}[e^{-r\tau}(P_\tau - S^i(P_\tau) - \theta)]$.

4. The winning bidder pays the initial cost X, invests rationally, and pays the contingent bid. In other words, if bidder i wins, she invests using stopping rule $\tau^i = \tau_{\theta_i}^i$ and makes the contingent payment.

Note that the seller's valuation $R(S^i)$ is not nessarily the same as what she gets in expectation $R_{\theta_i}(S^i)$. One may question if the setup of the game misses out any informal offers. For example, if the timing of investment is contractible, a bidder could offer to contract on that. The results are robust to additional side contracts as one can enlarge the security space from $S(P_t)$ to $S(\mathcal{I}_t)$ where $\mathcal{I}_t$ is the entire information set that is contractible, as long as limited liabilities hold. The proofs are exactly the same with minor changes in notations.

Notice that the optimal and efficient mechanisms are difficult to implement in practice according to the Wilson Doctrine, which argues for the avoidance of strong assumptions on the fine details of the bidders' distributions. Compare to other auctions using cash or standard securities, these mechanisms also suffer from the complexity and the requirement of single-dimensional types. In practice, potential bidders are unfamiliar with the optimal contract and may refrain from participating due to potential miscalculations, collusions, and seller's failures to commit to not to resell. As such, it is important to understand auctions with standard security bids as discussed earlier, and with a combination of cash and securities. The results about informal auctions are also robust to the combination of cash and securities, a topic of the next section.

5.2 Bidding Equilibrium in Informal Auctions

Taking the cash flow level at which the auction is held as given, there is an *essentially unique* bidding equilibrium. Moreover, equilibrium investments are socially efficient.

Lemma 5.1. *In a bidding equilibrium, a participating bidder i has $\tau^i_{\theta_i} = \tau^*_i$ where τ^*_i is the stopping time corresponding to the threshold strategy with investment trigger $P^*(\theta_i)$.*

The intuition is that if a bidder upon winning does not invest efficiently, she can always deviate to a bid that results in efficient investment, and offers weakly more to the seller to increase her marginal probability of winning without reducing the payoff upon winning.

Lemma 5.2. *Informal auctions only admit fully-separating equilibria.*

Because every bidder upon winning would invest efficiently, a better type has greater valuation and can separate herself from worse types. This also implies that no two bidders bid the same type of security.

Proposition 5.1. *There is an essentially unique equilibrium in an informal auction. This equilibrium is equivalent, in terms of allocation outcome, expected payoffs and post-auction investment, to a first-price cash auction. In particular, investment in the project is efficient.*

Basically in equilibrium, the bids are all cash-like. When mimicking a worse type, a better type finds it cheaper to use a security that is less sensitive to the true type or creates more social surplus, thus cash-like securities that ensure efficient investment are the most attractive. As a better type is indifferent from mimicking a marginally worse type in equilibrium, all bidders must be using cash-like securities.

5.3 Endogenous Timing of Informal Auctions

Since in equilibrium, the payoffs are equivalent to a first-price cash auction and FPA and SPA with cash generate the same revenue, the auctioneer's timing problem is equivalent to the strategic timing of a second-price cash auction when only she can initiate the auction. The value to type θ when the auction is held at P_0 is

$$W(P_0,\theta) = \left[\left(\frac{P_0}{P^*(\theta)}\right)^\beta \frac{\theta}{\beta-1} - X\right]\mathbf{1}_{\{\theta\leq\theta_c(P_0)\}}\mathbf{1}_{\{P_0\leq P^*(\theta)\}} + [P_0-\theta-X]\mathbf{1}_{\{\theta\leq\theta_c(P_0)\}}\mathbf{1}_{\{P_0\ >P^*(\theta)\}}$$

If only the seller can initiate the auction, by a similar argument as given in the optimal timing of optimal and efficient mechanisms, the solution involves a threshold strategy with auction trigger P_0 solving

$$\int_{\underline{\theta}}^{\hat{\theta}} N(N-1)f(\theta)F(\theta)[1-F(\theta)]^{N-2}[\beta X - (\beta-1)P^*(z(\theta)) + \beta z(\theta)]d\theta = 0 \tag{18}$$

and when X is sufficiently small that $\hat{\theta} = \overline{\theta}$, Appendix M shows the threshold is higher than the threshold for an efficient mechanism, thus the auction is delayed in general.

In real life, especially in M&As, we often see bidders initiating the auction. References here. To fully endogenize the auction timing in an informal auction, one has to consider the Bayesian equilibrium in the optimal stopping game among the seller and bidders. I assume any indifference in initiating the auction is resolved by initiating early.

Proposition 5.2. *In the Bayesian auction game with informal bids where both the seller and the bidders can initiate, bidders always initiate with threshold $P_I(\theta)$ monotone in θ that solves*

$$\int_{\theta}^{\overline{\theta}} \frac{d}{dP}\frac{W(P,\theta)-W(P,\theta')}{P^\beta}|_{P=P_I}(N-1)f(\theta')[1-F(\theta')]^{N-2}d\theta' = 0 \tag{19}$$

The intuition is that in equilibrium, seller updates the belief of distribution of types. If by cashflow level P the auction has not been initiated, the seller truncates the types to $[P_I^{-1}(P), \overline{\theta}]$, and times the auction to maximize the second highest valuation. Type $P_I^{-1}(P)$ times the aution to maximize the informational rent (difference between the her valuation and the second highest valuation) whose present value starts to decrease earlier than that of the second highest valuation as we increases the initation threshold. Therefore type $P_I^{-1}(P)$ would initiate earlier than the seller.

6 Combining Cash and Securities

The assumption thus far that the project is the only pleageable payment is well-motivated: in many auctions and sales, anecdotal evidence suggests that budget constraints are a serious concern: "set-aside sales" are available to small firms in timber rights auctions (Bergsten, Elliott, Schott, and Takacs (1987)); joint bidding in Outer Continental Shelf auctions is partially motivated by cash constraints (Hendricks and Porter (1992)); many national governments limit the length and size of mineral leases; the role of budget constraint is more complicated in more complex auctions, as demonstrated in the auction of "Personal Communication Services" licenses (Salant (1997)).

In fact, security bids became popular partially as a tool for mitigating such liquidity constraints. Bidding in a spectrum auction requires a substantial amount of cash, and fundraising through external capital markets takes time. The winning bidders have borrowed billions of dollars in the thrid generation (3G) European wireless spectrum auctions and installment payment agreements are usually used in the Federal Communications Commission (FCC) bandwidth auctions, both cases amount to essentially debt agreements.[22] Many of the winning bidders could not make the payments, leading to numerous litigations and reauctioning of licences. Commercial real estate sales usually entail bidders taking loans from banks. Mergers and acquisitions, in particular leveraged buyouts, frequently involve external financed bids. Although bidders' budgets are not modelled explicitly,[23] the fact that budget constraints are common in auctions with significant ramifications justifies our focus on pure security bids thus far.

That being said, in many cases the liquidity constraints are not binding and the auctioneer can effectively utilize a combination of cash (investment-independent payments) and securities in designing the auctions, which allows the auctioneer greater flexibility in designing the selling mechanism. This section first discusses entry fee/reserve price, a common tool used in auctions, then examines how liquidity constraints impact optimal and efficient mechanisms, before finally relating to the opening story to examine bonus bid auctions - the predominant way of selling oil/gas leases.

[22]Please see Landler (1997). Bulow, Levin, and Milgrom (2009) give finer depiction of the complexity in large spectrum auctions.

[23]As more elaborate models of bidders' budgest, Che and Gale (2000) examine the optimal mechanism for selling to a budget-constrained buyer; Rhodes-Kropf and Viswanathan (2005) highlight the role of external financing for the bidders with cash constraints. Che and Gale (1996, 1998) explore the impact on revenue due to a combination of budget constraints and auction format.

6.1 Entry Fee/Reserve Price

In the private value auctions literature, both entry fee and reserve price operate on the exclusion principle, since they imply, in effect, that it is optimal for the seller to exclude some bidders even though their values exceed the seller's value of the project. In our setting, reserve price can be interpreted a a minimum cash payment at the time of allocation in addition to the security bids. Since to every reserve price, there is a corresponding entry fee excluding the same group of bidders, the subsequent discussions focus on entry fees.[24]

The main effect of entry fee is that the entry decision now depends on the amount of competition, in addition to the real option value of the project. Absent entry fee, $\hat{\theta}$ is independent of N. A bidder enters when the real option value of the project is non-negative. When there is entry fee, the probability of winning has to be taken into consideration. Appendix N establishes the following result:

Proposition 6.1. *With the same entry fees, FPAs and SPAs of real options with cash bids and security bids generate the same cutoff type for participation, which is decreasing with greater competition. Moreover, cash-bid FPAs and SPAs yield the same expected revenue and social welfare.*

Also shown in the Appendix, Propositions 2.1, 2.2 and Corollary 2.1 still hold with the new cutoff type.

Entry fee or reserve price in auctions with security bids, similar to those in traditional auctions, is a useful tool for enhancing the seller's revenue, as the next proposition shows.

Proposition 6.2. *In SPAs with security bids, a revenue-maximizing seller always sets an entry fee. This also holds for FPAs if the expected present value of the security a type pays is non-decreasing in* $\hat{\theta}$.

The intuition is that by setting a small entry fee, we exclude a marginal bidder from participating. However the revenue from this marginal bidder is zero because this person is indifferent between participating and the outside option of zero. Yet there is a positive measure of bidders participating, the fee collected is positive. In FPAs this is complicated

[24] As in DeMarzo, Kremer, and Skrzypacz (2005), there is no entry cost here. But the analysis readily applies to auctions with entry costs, which would induce endogenous entries not caused by auction design. The only difference is that the entry cost is not a wealth transfer, but a social waste. The qualitative implications remain.

with the fact that changing the cutoff type also changes the bidding strategies for all types, and the expected present value of the security a type pays being non-decreasing in $\hat{\theta}$ is a sufficient condition for the result. Numerical simulations also indicate that charging an entry fee could be welfare-improving, a feature absent from the literature of independent private value auctions.

In addition, being able to set an entry fee allows the auctioneer to effectively sell projects with very small initial commitment cost $X \to 0$, for which FPAs and SPAs with pure security bids admit no unique equilibrium, if at all. To see this more clearly, in FPAs, the cutoff type is the worst type. In any candidate monotone equilibrium, this type has zero chance of winning, thus would increase her bid as long as the payoff from the project is positive, in order to increase the chance of winning and getting a positive payoff from participating. In SPAs, every type bids up to their valuation. Thus the worst type in FPAs and all types in SPAs would increase s until $\max_\tau[e^{-r\tau}(P_\tau - S(s, P_\tau) - \theta)] = 0$. In equity auctions this means everyone is bidding 100% shares, resulting in a degenerate bidding equilibrium.[25]

While this problem limits the auctioneer's ability to use simple auction formats with standard securities to sell the project, an entry fee or reserve price comes to rescue. By charging an entry fee $c > 0$, there is still no pure equilibrium bidding strategy in SPAs because the entry fee is sunk upon entering the auction, the bidders always want bid till the option value is zero, yet the real option has strictly positive values to them, thus they would potentially delay the exercise indefinitely. This contradicts the rationality of entering the auction in the first place. But if no one enters, one can always be better off by entering and offering no securities. Nevertheless equilibria exist for FPAs if (R1) is satisfied, and often lead to closed-form solutions for the bidding strategies, which is extremely useful in practice when the auctioneer tries to estimate the revenue. For example, in equity auctions where there is inherent option value to wait for every bidder if she were to own the project entirely ($P_0 \leq \frac{\beta}{\beta-1}\underline{\theta}$), as $X \to 0$, the bidding strategy is characterized by $\alpha'(\theta) = -\frac{(N-1)f(\theta)(1-\alpha)}{\beta(1-F(\theta))}$, whose general solution in closed form is

$$\alpha(\theta) = \int_{\theta}^{\hat{\theta}} \frac{(N-1)f(\theta'')}{\beta(1-F(\theta''))} e^{\int_{\theta''}^{\theta} \frac{(N-1)f(\theta')}{\beta(1-F(\theta'))} d\theta'} d\theta'' \tag{20}$$

[25] For this reason, an initial commitment cost is common in the security-bid auctions literature. For example, see DeMarzo, Kremer, and Skrzypacz (2005) and Che and Kim (2010). Rhodes-Kropf and Viswanathan (2000) also find that without this auction friction, the equilibrium in equity auctions is non-separating.

Distributions	Uni. $F(\theta)=\frac{\theta-\underline{\theta}}{\overline{\theta}-\underline{\theta}}$	Exp. $F(\theta)=1-e^{\lambda(\theta-\underline{\theta})}$	Pareto $F(\theta)=1-(\frac{\underline{\theta}}{\theta})^{\gamma}$
Equity FPAs	$\alpha(\theta)=1-(\frac{\overline{\theta}-\hat{\theta}}{\overline{\theta}-\theta})^{\frac{N-1}{\beta}}$	$\alpha(\theta)=1-\exp\{-\frac{\lambda(N-1)(\hat{\theta}-\theta)}{\beta}\}$	$\alpha(\theta)=1-(\frac{\theta}{\hat{\theta}})^{\frac{(N-1)\gamma}{\beta}}$

Table 2: Bidding Strategies in Equity FPAs for Common Distributions

with $\alpha(\hat{\theta})=0$ for $\theta \leq \hat{\theta}$. Table 2 summarizes the solutions for some common distributions.

6.2 Optimal and Efficient Mechanisms

Let the investment-independent payment to the auctioneer be $L(\tilde{\theta}_i,\theta_{-i})$. If $L(\tilde{\theta}_i,\theta_{-i})>0$, it should not happen after the exercise, as the buyer can simply walk away without paying. If $L(\tilde{\theta}_i,\theta_{-i})<0$ (subsidy to the bidder), there is an equivalent upfront subsidy. Therefore we only need to consider the payments occuring before the exercise of the option. We note that this could still be contingent on some processes other than P, but for now, we can simply view it as an up-front cash payment to the seller at the time of the contract. Let L_i denote bidder i's liquidity constraint and L_0 the auctioneer's. The space for possible mechanisms now includes any combination of cash and securities.

Lemma 4.1 remains intact while Propositions 4.1 and 4.2 are modified in that the combination of liquid payment L and security payment S satisfies (14). Appendices H and I characterize all optimal and efficient mechanisms with varying degrees of liquidity constraints. In general, there is a continuum of them involving various combinations of cash and security payments. As in Board (2007b), when the liquidity constraints are not binding, one particular optimal mechanism entails a security payment $S(\theta)=F(\theta)/f(\theta)$ which is time- and state- invariant, and the auctioneer does not need to verify post-auction information to implement it. An implementation using FPA in upfront cash is given in the appendix. Similarly, there is an efficient mechanism that involves no security payment. These mechanisms correspond to the solution in Board (2007b). The multiplicity of optimal and efficient designs highlights the complementarity of liquidity and investment-contingent contracts: if it is hard to monitor post-auction information for a particular bidder (this is especially true when the P evolves differently for different bidders), the auctioneer can require a larger liquid payment; if a bidder is too liquidity-constrained, the auctioneer can contract on the investment timing to achieve the maximum revenue.

Since there are many optimal and efficient mechanisms, a natural question is whether it is possible to find efficient mechanisms that yield higher revenue than cash auctions. While this is possible without post-auction moral hazard as demonstrated in the literature of security-bid auctions, it is a futile attempt with post-auction moral hazard. Among all efficient mechanisms, the revenue is the difference between the total welfare and the winning bidder's profit. The latter differs by at most a constant according to (14). Since cash auctions involve the same cutoff type whose utility is zero, thus

Corollary 6.1. *FPAs and SPAs with cash bids implement the efficient mechanism with the highest revenue.*

This is why "inferred profit" auctions in Section 3 do not yield higher revenue than cash auctions. Similarly,

Corollary 6.2. *All optimal mechanisms yield the same welfare.*

6.3 Bonus-bid Auctions

When the bidders are not liquidity-constrained, the security design could be very flexible. In recent years many oil and mineral lease auctions by the government involve fixing a royalty rate and letting the contractors bid in up-front payments called the "bonus". For onshore federal leases, the Minerals Lands Leasing Act prescribes the base share of royalty rate at 1/8 the value of production; for offshore leases, the Outer Continental Shelf Lands Act sets the base royalty rate at 1/6 the value of production.

Suppose the seller sets the royalty rate α and let the buyers compete via bonus bids. From earlier discussion on royalty auctions, in a second price auction every participant bids up to the expected value $L(\theta) = (\beta-1)^{\beta-1}\beta^{-\beta}P_0^{\beta}(1-\alpha)^{\beta}\theta^{1-\beta} - X$, which is decreasing in α. $L(\hat{\theta}) = 0$ gives the cutoff type. The expected revenue to the seller would be this bonus bid plus the contingent part which is

$$S(\theta_{(1)}) = D(P_0; P_t^*(\theta_{(1)}))\alpha P_t^*(\theta_{(1)}) = P_0^{\beta}(\beta-1)^{\beta-1}\alpha(1-\alpha)^{\beta-1}\beta^{1-\beta}\theta_{(1)}^{1-\beta}$$

The expected total revenue is simply $R_{bonus} = \mathbb{E}[L(\theta_{(2)})1_{\{\theta_{(2)}<\hat{\theta}\}} + S(\theta^{(1)})1_{\{\theta_{(1)}<\hat{\theta}\}}]$. The cutoff type, revenue and social welfare in the FPA bonus-bid auctions are the same because this is basically a cash auction for $(1-\alpha)$ fraction of the project and revenue equivalence between FPA and SPA applies.

It can be verified that the realized option value is a fraction $(1-\alpha+\alpha\beta)(1-\alpha)^{\beta-1}<1$ of the efficient value, and it is decreasing in α. While when $X \to 0$, either $\beta \to 1$ or $\alpha \to 0$ would result in a revenue higher than cash auctions, bonus-bid auctions do not yield higher revenue in general. The next section discusses the implications of royalty or bonus auctions on the timing in investments, which could potentially explain why most of the oil and gas leases are idle despite that fact that it is socially efficient to develop them.

7 Model Implications

The model has three main implications. First, this paper's focus on the post-auction moral hazard associated with real options provides a natural framework to study investment timing, one of the most important concepts in the theory of corporate investment, and its relation to selling mechanisms. Second, conventional wisdom in auctions says increasing the competition among potential bidders enhances expected revenue and social welfare when allocation is efficient, security ranking tends to be universal and independent of number of bidders and auction timing, and security bids often dominate cash bids. These no longer hold for auctions of real options. Third, strategic timing of auctions affects expected revenue, welfare and security ranking, and should be an important part of the auction design.

7.1 Inefficiencies in Investment Timing

Earlier sections have already demonstrated that investments are delayed in equity auctions, bonus auctions and friendly debt auctions, and accelerated in call option auctions. Even the revenue maximizing mechanism delays investments. Fig. 1 gives an illustration of the differences in timing. From this simulation, the investment could occur at very different times depending on the selling mechanism used. Using the properties of Wald distribution, the time delay t_D when the investment threshold is increased from P_1 to P_2 is described by,[26]

$$f_{GBM}(t_D; P_1, P_2) = \frac{\ln \frac{P_2}{P_1}}{\sqrt{2\pi\sigma^2 t_D^3}} \exp\left[-\frac{\left[\ln \frac{P_2}{P_1} - \left(\mu - \frac{\sigma^2}{2}\right) t_D\right]^2}{2\sigma^2 t_D} \right] \tag{21}$$

[26]See Taylor and Karlin (1975) pg.363, and Chhikara and Folks (1989).

with the mean and shape parameter given by $m = \frac{\ln\left(\frac{P_2}{P_1}\right)}{\mu - \frac{\sigma^2}{2}}$ and $y = \left(\frac{\ln\left(\frac{P_2}{P_1}\right)}{\sigma}\right)^2$. In the case of royalty auctions or bonus-bid auctions for oil leases, the expected investment lag is thus,

$$\Gamma = \frac{-\ln(1-\alpha)}{\mu - \frac{\sigma^2}{2}} \tag{22}$$

where we have assumed $\mu - \frac{\sigma^2}{2} > 0$ for the expectation to exist.[27]

Next, I examine the effects of changing the underlying parameters on the expected investment lags. For bonus-bid auctions $\alpha > 0$ is fixed, and when the lease is sold, there is always going to be development delays. $\frac{\partial \Gamma}{\partial \alpha} > 0$, $\frac{\partial \Gamma}{\partial \mu} < 0$ and $\frac{\partial \Gamma}{\partial \sigma} > 0$, thus only the royalty rate, drift and diffusion parameters affect the expected development delays, N, X, P_0 and r do not. In addition, $\frac{\partial^2 \Gamma}{\partial \alpha^2} > 0$, $\frac{\partial^2 \Gamma}{\partial \sigma^2} > 0$, $\frac{\partial^2 \Gamma}{\partial \sigma \partial \alpha} > 0$. Not only do more volatile times or high royalty rates result in longer delays, but they are mutually reinforcing and with increasing marginal effects.

For royalty auctions, the effect is more complicated. For example, consider the effect of market volatility in SPA with royalty bids. First, using the fact $\frac{\partial \beta}{\partial \sigma} < 0$, $\hat{\theta} = \theta_c$,

$$\frac{d\hat{\theta}}{d\sigma} = \frac{\partial \hat{\theta}}{\partial \beta}\frac{\partial \beta}{\partial \sigma} = \frac{\theta_c}{(\beta-1)^2} \ln \frac{\beta X}{P_0} \frac{\partial \beta}{\partial \sigma} > 0 \tag{23}$$

thus a higher volatility increases the probability that $\alpha > 0$, thus the chance of an investment delay. Conditional on there being delay, i.e., $\theta_{(2)} < \hat{\theta}$,

$$\frac{\partial \Gamma}{\partial \sigma} = \frac{\sigma \ln(1-\alpha)}{\mu - \frac{\sigma^2}{2}} + \frac{1}{\mu - \frac{\sigma^2}{2}} \frac{\partial \beta}{\partial \sigma} \frac{\partial}{\partial \beta} \ln \left[\frac{\beta^\beta X \theta_{(2)}^{\beta-1}}{P_0^\beta (\beta-1)^{\beta-1}} \right]^{\frac{1}{\beta}} \tag{24}$$

The first term is positive and simply reflects the fact that it takes longer in general for the Geometric Brownian Motion to first hit a target threshold. The second term is also positive $\frac{\partial}{\partial \beta} \ln \left[\frac{\beta^\beta X \theta_{(2)}^{\beta-1}}{P_0^\beta (\beta-1)^{\beta-1}} \right]^{\frac{1}{\beta}} = \frac{1}{\beta^2} \ln \left[\frac{X(\beta-1)}{\theta_{(2)}} \right] < 0$. It is a well-established result that increasing

[27] If $\mu < \sigma^2/2$, the median lag M can be considered instead. It satisfies $\Phi\left[\frac{\ln(1-\alpha)+(\mu-\sigma^2/2)M}{\sigma\sqrt{M}}\right] + (1-\alpha)^{\frac{\sigma^2-2\mu}{\sigma^2}} \Phi\left[\frac{\ln(1-\alpha)-(\mu-\sigma^2/2)M}{\sigma\sqrt{M}}\right] = \frac{1}{2}$, and numerical simulations yield the same qualitative results under a wide range of parameters.

uncertainty leads to increased real option value.[28] Since each bidder increases the bid α until she breaks even, a higher real option value means a higher bid, which in turn would imply a higher investment threshold compared to the efficient level. The comparative statics with respect to other parameters are summarized in Table 3. FPAs can be similarly analyzed. Notice also that the time lag is big when the auction is held at a higher cash flow level or when the auction friction X is small. The intuition is that when P_0 is big or X is small, the undertaking of the project is more attractive and the bidders bid more aggressively, which further distorts the incentive to invest efficiently.

Increase in Parameter	σ	μ	r	X	P_0
Probability of Delays	Increase	Increase	Decrease	Decrease	Increase
Expected Delay (if happens)	Increase	Increase	Ambiguous	Decrease	Increase

Table 3: Impact of Changing the Fundamental Parameters in SPA with Royalty Bids

This is consistent with empirical observations. The US Department of the Interior experimented with royalty and profit-rate auctions from 1978-1983, in which the government fixed a small up-front "bonus" payment, and allows the bidders compete in royalty rates. Many bidders thus bid extremely high royalty rates in order to win the auctions. In the end, most fields were simply not developed, even when it was economically efficient to do so. The oil price level and volatility were extremely high during that period, and the significant delays are consistent with the comparative statics. Even as many auctions of natural resources converged to FPAs with bonus bids in recent years, the investments can be inefficiently delayed as shown. In reality, many other strategic interactions among the bidders complicate the issue. For example, Beshears (2011) shows alliances in oil and gas drilling perform better than solo bidders; Hendricks and Porter (1996) attributes the delays in exploratory drilling to free-rider problem and war of attrition. The above analysis, though stylized, is meant to complement rather than contradict these other factors. By highlighting the destructive powers of moral hazard in investment timing, this paper suggests a potential explanation for why the Obama administration finds large tracts of federal lands remaining idle.

The Department of the Interior is currently exploring policy options to provide companies with additional incentives for more rapid development of oil and gas resources from existing

[28]For example, seeMcDonald and Siegel (1986).

and future leases.[29] Without prescribing detailed policy changes, this paper suggests the following: bonus-bid auctions necessarily cause investment delays due to misalignment of incentives in lease contracts. Hence there is always the trade-off between social efficiency and revenue extraction. Any useful policy recommendations should first focus on reducing the post-auction moral hazard that is inimical to both the revenue and social welfare. For example, ensuring the contractibility of P would help mitigate the moral hazard. Moreover, as discussed in the remaining sections, the timing of the auction, the number of bidders admitted, and the initial cost of exploration could all impact the severity of the post-auction moral hazard, and should be taken into consideration when designing the auctions.

7.2 Nonconventional Auction Design

Many standard results in the auctions literature may not hold with security bids and post-auction moral hazard, leading to nonconventional auction designs.

Auctions versus Negotiations

It is a well-established result in independent private value auctions that increasing the number of bidders enhances competition and the seller's revenue, and potentially social welfare. The importance of competition is perhaps best articulated in Bulow and Klemperer (1996):

"*With independent signals and risk-neutral bidders, an absolute English auction with* $N+1$ *bidders is more profitable in expectation than any negotiation with* N *bidders.*"

But in reality sellers often restrict the number of bidders. For example, private companies and divisions of public companies are often sold through an auction process, where the advisor of the selling company acts as de facto auctioneer. Although sometimes it pays to restrict bidders by charging entry fees, it is puzzling that these corporate auctions restrict the number of bidders absent entry fees (see Hansen (2001) and French and McCormick (1984)). The moral hazard associated with security bids provides a potential explanation.

Basically as N increases, the competition intensifies, and the security the winner is paying is increased. The resulting increase in moral hazard could outweight the benefit of a better realized type for the winner. Combining with Prop. 6.2, there are thus circumstances that

[29]Recent measures include increasing rental rates, tiering durational terms and increasing minimum bids, according to the updated report by U.S. department of the Interior.

the seller does better charging an entry fee or reserve price with N bidders, rather than having a free-entry-no-reserve-price auction with $N+1$ bidders. There is no contradiction, however, because the traditional auctions literature rarely considers post-auction moral hazard that in many cases is rather damaging to both the seller's revenue and the social welfare.

A complete characterization of the impact of increased competition on welfare and revenue is extremely challenging due to the compl exity of real options and stochastic post-auction environment. Instead, numerical simulations are given in the same spirit as in Samuelson (1985) to indicate that revenue and welfare could vary almost in any way with N. In Fig. 2, expected revenues and welfares in SPAs with equities are plotted against N. While (a) Increasing Welfare and (d) Increasing Revenue are intuitively expected, (b) Decreasing Welfare, (c) Non-monotone Welfare with global maximum 6.38 at $N = 11$, (e) Decreasing Revenue and (f) Non-monotone Revenue with global maximum 8 at $N = 9$ give the surprising result that as N increases, the cost of moral hazard could dominate the revenue or welfare improvement due to increased competition.

The result generalizes to auctions with friendly debts and call options too (see Figure 3), and is quite robust to distributional assumptions and endogenous entries with entry cost/fee based on simulations with other distributions such as Pareto distributions and exponential distributions. Depending on the parameters, the expected social welfare and revenue to the seller need not increase with the number of potential bidders. Thus, policies to limit the number of potential bidders may be revenue- or welfare-improving.

Security Ranking

A well-known result in DeMarzo, Kremer, and Skrzypacz (2005) is that steeper securities yield higher expected revenues. It is no longer the case in the presence of adverse selection and moral hazard. Che and Kim (2010) demonstrate adverse selection effect could lead to the opposite ranking. Kogan and Morgan (2010) compares equity and debt when an entrepreneurs bid for VC's resources, and show the revenue ranking depends on the returns to entrepreneurial effort. In the current setting, the ranking is very parameter-sensitive. The general observation is that equity and call options usually dominate friendly debt. Increasing P_0, or N or X makes it more likely for call option to dominate equity auction, but increasing c makes equity auction more likely to dominate call option auction. Here β, N, X, P_0 all matter, and the relationship is far more complex: in Fig. 3, the ranking is sensitive to the number of potential bidders while in 4, the ranking is sensitive to the cash flow level at which auctions are held. In both figures, though equity generally dominates friendly debt, it could either dominate or be dominated by call options despite being less "steep". These results are robust to the introduction of entry fee or entry cost and are novel in linking security design to the level of competition and the timing of auction.

While a general ranking of securities are highly parameter dependent, prior studies indicate that security bids usually perform better than cash bids. This would not be the case as the number of bidders increase, as discussed next.

Cash versus Standard Securities

Rhodes-Kropf and Viswanathan (2000) show that any securities auction generates higher expected revenue to the seller than a cash auction. However, with post-auction moral hazard, the security ranking is dependent on the number of bidders. Since the linkage advantage of security bids lies in the extraction of the winning bidder's rent, it decreases in expectation when N becomes big. But moral hazard persists with many standard securities which reduces revenue and welfare. Therefore, it may not be optimal to exploit the linkage advantage of security bids when there are many potential bidders. In particular, cash could generate higher revenue than many standard securities such as equities.

First note that any standard security $S(s,P)$ can be approximated by $\sum_{i\in I} a_i(s)[P - b_i(s)]^+$, where I is a countable set and $\sum_i a_i(s) \leq 1 \forall s$ to ensure $P - S(s,P)$ is weakly increasing in P. Suppose $\underline{s}$ is the security the type $\underline{\theta}$ bids, without loss of generality $b_i(\underline{s}) \leq b_j(\underline{s})$ if $i \leq j$. I define the following class of security:

DEFINITION A *regular security* is a standard security for which the above approximation is exact, and for $\frac{\beta}{\beta-1}\underline{\theta} \in [b_m, b_{m+1})$,

$$\min\left(\left|\frac{\beta}{\beta-1}\underline{\theta} - b_m\right|, \left|\frac{\beta}{\beta-1}\underline{\theta} - b_{m+1}\right|, \left|\sum_{i\leq m} a_i b_i - \underline{\theta}\sum_{i\leq m} a_i\right|\right) > M, \tag{25}$$

for some $M > 0$.

The conditions basically imply that moral hazard is present, as shown in Appendix P. In fact most common securities are regular securities. For example, equity has $a_1 = \alpha(\underline{\theta})$,

$a_2 = b_1 = 0$, $b_2 = \infty$. Thus $b_m = 0$, $b_{m+1} = \infty$, and

$$\min\left(|\frac{\beta}{\beta-1}\underline{\theta} - b_m|, |\frac{\beta}{\beta-1}\underline{\theta} - b_{m+1}|, |\sum_{i\leq m} a_i b_i - \underline{\theta}\sum_{i\leq m} a_i|\right) = \min(\frac{\beta}{\beta-1}\underline{\theta}, \underline{\theta}\alpha(\underline{\theta})) > 0 \quad (26)$$

as in both FPA and SPA $\alpha(\underline{\theta}) > 0$. Similarly, call option and friendly debt are regular securities too.

Proposition 7.1. *As the number of potential bidders gets large, cash bids dominate regular securities in FPAs and SPAs in terms of expected revenue and social welfare.*

In particular, cash bids dominate equity bids, call option bids and friendly debt bids. Since the class of regular securities include many commonly used securities, the auctioneer has to consider the size of the bidders market in order to choose the desirable security. The result also agrees with empirical observations that security bids are seldomly used when the number of bidders is large.

Of course, other considerations also influence security design. For example, it is well-known that the medium of exchange relates to private information and acts as signals to the market.[30] This discussion simply shows that moral hazard alone could also lead to preference for cash in large bidders markets. Gorbenko and Malenko (2011) give another situation that cash dominates equity bids when the number of sellers and the corresponding bidders' markets are large.

7.3 Strategic Timing of Auctions

The strategic timing of auctions is an important yet understudied topic for the sales of assets and projects whose values evolve over time. Real life examples are prevalent: realtors sell foreclosed house under specific housing market conditions, acquirer firms strategically initiate the bidding for target firms in an M&A deal, *e.t.c.*. Proposition 2.3 shows cash flow level at which the auction is held can fundamentally impact bidding behaviors. Propositions 4.3-4.4 also demonstrate how strategic timing is a crucial factor in the optimal and efficient mechanism designs, and in informal auctions.

[30] Eckbo, Giammarino, and Heinkel (1990), Rappaport and Sirower (1999), and Betton, Eckbo, and Thorburn (2009) are some examples. Malmendier, Opp, and Saidi (2011) is a recent study showing the advantages of using cash in M&A.

The effect of timing applies to auctions with standard security bids as well. For any security bid, while post-auction moral hazard lies in the timing of the investment, its magnitude depends crucially on the timing of the auction itself. Because of this, unlike the case of optimal mechanisms or informal auctions, auctions could be accelerated relative to the socially optimal. The optimal timing of auctions with standard securities depends on the specific securities used, Figures 4-6 illustrate how timing the auction can yield higher revenue or welfare.

There are two key takeaways: first, security ranking is sensitive to the auction timing. As shown in Figure 4, at $P_0 = 280$ cash gives the highest revenue of 3.3, among the contingent securities, equity gives the highest of 3.2 and call option the lowest at 2.1; at $P_0 = 360$, cash gives 4.8, call option is the highest among the three contingent securities at 3.8, debt the lowest at 1.9. This is observed in FPAs too: in Figure 5, at $P_0 = 240$ equity dominates cash whereas at $P_0 = 300$ cash dominates equity. The second takeaway is that strategic timing could be more important than security design. In Figure 4, whether the bidders are cash-contrained or not, the worst security design at $P_0 = 300$ outperforms the best security design at $P_0 = 220$ by at least 1.5 (more than double the revenue). The results hold with other common distributions too.

There could be factors exogenous to the model that affects the strategic timing of auctions. For one, delaying the auction risk creating market uncertainty and delaying the introduction of new technology or development, and potentially losing the "first-mover" advantage. The sale of the British 3G telecom licences serves an excellent example. Despite the dilemma faced by the government because of the hostile takeover attempt of Mannesmann-Orange by Vodafone[31], the auction team was concerned that delaying the auction would delay the introduction of 3G, and create uncertainty of profitability of the technology. Moreover, subsequent 3G auctions might be less competitive because of reduced entry and alliances among bidders. The prices in the first auction might be driven higher if bidders believe winning this auction gives a competitive advantage in future auctions. These concerns all turned out to be true in later 3G auctions in Austria, Belgium, Denmark, Germany, Greece, Netherlands and Switzerland.[32] Another motivation for timing the auction is the coordination of market players. Auctions of futures contracts on electricity provision is a good example. The construction of facilities for generating and delivering electricity takes time

[31] For details, please see Binmore and Klemperer (2002).
[32] See section 7.3, Klemperer (2002).

and an early auction allows the winning bidder ample time to prepare for the undertaking.

Despite these possibilities, this paper shows that understanding the strategic timing of auctions is important. Gorbenko and Malenko (2013) have already demonstrated how the timing of the auction significantly impacts merger payments and terms. More research should follow to help understand strategic timing in broader contexts.

8 Extensions and Discussions

8.1 Robustness of Mechanisms

In mechanism theory, many recent studies have emphasized the importance of robustness. [References here.] A regret proof mechanism is in many ways easier to implement. In the curret setting, the allocation of the project is ex-post optimal in optimal and efficient mechanisms, and in informal auctions. However, while formal auctions with security bids lead to monotone bidding equilibria, they also allow allocations to be potentially sub-optimal ex post based on the auctioneer's inference of the bidders' types. The same robustness concern appears in Che and Kim (2010).

The question, then, is why in practice people use auctions with standard securities. There are potentially two explanations. First, a robust mechanism may be hard to implement due to the complexity and strong requirement on prior knowledge. The bidding rules for the standard securities are easy to follow. This agrees with the spirit of formal auctions where allocation rules are fait accompli, and the auctioneer commits to them. Another reason is that the allocations in auctions with standard securities are often optimal ex post, such as those in SPAs with equity bids, or in FPAs with equity bids where even the best type would not bid too many shares. Another way to ensure robustness is to restrict the range of bids $[s_L, s_H]$. This may lead to partial pooling at the top, but would not cause post-auction regret.[33]

8.2 Options with Expirations

So far we have assumed that the investment option is perpetual. This is a reasonable simplification because investment decisions in practice often involve short time scales relative

[33] A previous version of the paper examines this, and the analysis shows the results are robust to this restriction.

to the time window of investment opportunity. For example, the duration for land leases are typically 30-50 years and constructions are often planned and implemented in a few years. However, it is useful to understand how the results are modified as a real option approaches expiration. Moreover, many investment options lose value due to unforeseen circumstances such as natural disasters and regulatory reforms. Such circumstances are best modeled with stochastically arriving expirations. This section explores implications both stochastic and deterministic expirations.

Stochastic Expiration

Suppose with arrival intensity λ, the investment option is rendered worthless. This simply augments the discount rate by λ. [More details to be added. The comparatives w.r.t. r are more relevant now.]

Deterministic Expiration

[More details to be added] With deterministic expiration time T, the project is equivalent to an American call option on a stock whose price process follows P_t, and pays a stream of dividend $(r - \mu)P_t$. Though not analytically tractable, numerical results show the general results hold. [Details to be added]

8.3 Common Values and Afflicated Values

[to be added]

8.4 Entry Costs

Potential bidders may incur cost of due dilligence, which can be modeled as an entry cost. The analysis is very similar to the case of entry fee, save for the fact that this is a social cost, not a wealth transfer.

9 Conclusion

This paper analyzes the auctions of real options in various formats, including cash bids, contingent claim bids, and a combination of both. The post-auction moral hazard in investment timing associated with security-bids are studied in details and the implications include domination of cash-bid in large bidders markets, decreasing revenue and social welfare with increased number of potential bidders, and dependence of security ranking on the market size and auction timing. This paper is also the first to introduce strategic timing as an optimal stopping problem for the auctioneer, and is one of the first to characterize the initiation of informal auctions.

Admittedly, the model in this paper is stylized and only partially captures the complexity of real investment options and security-bid auctions. For example, bidders for oil leases differ in more dimensions and the development of an oil field involve multiple dynamic decisions instead of an one-shot investment. The quantity of investment in addition to the time of investment is crucial too. I also leave out resale of the project and renegotiations. So far the investment cost signal θ is also assumed to be exact. Bidders oftentimes cannot ascertain the investment costs or other unknown variables of the project until the project is in their hands, and they may acquire additional information post-auction. This is definitely the case for oil leases where the bidders can only conduct seismic studies yet the winning bidder can carry out more exploratory drills. In the auctions for the foreclosed houses, the bidders are not allowed to go into the house, and the winning bidder acquires much more information after the auction about the house. Nonetheless, the analysis concerning the dynamics of auctions and investments may find useful applications in more specific contexts and add insights to the literature on auctions and contracting in the real options framework.

This paper cautions the use of security-bid auctions as they become an increasingly useful tool for both government and industry: the security design has significant impacts on the exercise of real options, and the post-auction moral hazard should be taken into account when deciding on the selling mechanisms. Moreover, auctioneer (and bidders) could strategically time the auction or informal negotiation, which significantly impacts bidding equilibrium, auction payoffs, and post-auction investments.

More work is clearly needed on this academic front. In particular, applications of the framework to explain specific economic phenomina should prove fruitful. One example is auctions of oil leases with common value consideration and greater institutional details incorporated into the analysis. Selling mechanisms with noisy cost signals potentially entails sequential screening and also constitutes an interesting extension of the current study. Finally, auction timing with resales is also worth exploring. The real challenge lies in developing these research options in a timely and responsible manner.

References

Bergsten, C.F., K.A. Elliott, J.J. Schott, and W.E. Takacs, 1987, *Auction Quotas and United States Trade Policy* (Institute for International Economics, Washington, DC).

Beshears, J., 2011, The performance of corporate alliances: Evidence from oil and gas drilling in the gulf of mexico, .

Bessy, C., and E. Brousseau, 1998, Technology licensing contracts features and diversity1* 1, *International Review of Law and Economics* 18, 451–489.

Betton, S., B.E. Eckbo, and K.S. Thorburn, 2009, Merger negotiations and the toehold puzzle, *Journal of Financial Economics* 91, 158–178.

Binmore, Ken, and Paul Klemperer, 2002, The biggest auction ever: The sale of the british 3g telecom licences, *The Economic Journal* 112, C74–96.

Board, Simon, 2007a, Bidding into the red: A model of post-auction bankruptcy, *Journal of Finance* 112, 2695–2723.

——— , 2007b, Selling options, *Journal of Economic Theory* 136, 324–340.

Bulow, J., and P.A.L. Klemperer, 1996, Auctions versus negotiations, *The American Economic Review* 86, 180–194.

Bulow, J.I., J.D. Levin, and P. Milgrom, 2009, Winning play in spectrum auctions, .

Caves, Richard E., 2003, Contracts between art and commerce, *Journal of economic Perspectives* 17(2), 73–83.

Che, Y.K., and I. Gale, 1996, Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints, *Economics Letters* 50, 373–379.

——— , 1998, Standard auctions with financially constrained bidders, *The Review of Economic Studies* 65, 1.

——— , 2000, The optimal mechanism for selling to a budget-constrained buyer* 1, *Journal of Economic Theory* 92, 198–233.

Che, Y.K., and J. Kim, 2010, Bidding with securities: Comment, *American Economic Review* 100, 1929–35.

Chhikara, R.S., and L. Folks, 1989, *The inverse Gaussian distribution: theory, methodology, and applications* . vol. 95 (CRC).

Chisholm, Darlene C, 1997, Profit-sharing versus fixed-payment contracts: Evidence from the motion pictures industry, *Journal of Law, Economics, and Organization* 13, 169–201.

Choi, J.P., 2001, Technology transfer with moral hazard, *International Journal of Industrial Organization* 19, 249–266.

Choquet, Gustave, 1966, *Topology* (Academic Press New York and London).

Cohn, Donald L., 1994, *Measure Theory* (Birkhuser).

Cong, Lin William, 2012a, Auctions with contingent payments - just the beginning, *(Preliminary Working Paper).*

——— , 2012b, Costly learning and agency conflicts in investments under uncertainty, *(Working Paper).*

Crémer, Jacques, 1987, Auctions with contingent payments: Comment, *American Economic Review* 77, 746.

Datar, S., R. Frankel, and M. Wolfson, 2001, Earnouts: The effects of adverse selection and agency costs on acquisition techniques, *Journal of Law, Economics, and Organization* 17, 201.

DeMarzo, Peter, Ilan Kremer, and Andrzej Skrzypacz, 2005, Bidding with securities: Auctions and security design, *American Economic Review* 95(4), 936–959.

DeMayo, Peter, 1983, Bidding on new ship construction, *R. Engelbreciit-Wiggans et al* pp. 371–87.

Dessauer, John P., 1981, *Book Publishing* (NY: Bowker).

Dixit, A. K., and R. S. Pindyck, 1994, *Investment Under Uncertainty* (Princeton University Press, Princeton, NJ.).

Duffie, D., 2009, *Dynamic Asset Pricing Theory* (Author's Manuscript, Princeton university Press).

Eckbo, B.E., R.M. Giammarino, and R.L. Heinkel, 1990, Asymmetric information and the medium of exchange in takeovers: Theory and tests, *Review of Financial Studies* 3, 651–675.

Engelbrecht-Wiggans, R., 1987, On optimal competitive contracting, *Management science* pp. 1481–1488.

French, K.R., and R.E. McCormick, 1984, Sealed bids, sunk costs, and the process of competition, *Journal of Business* pp. 417–441.

Gorbenko, A.S., and A. Malenko, 2011, Competition among sellers in securities auctions, *The American Economic Review* 101, 1806–1841.

Gorbenko, Alexander, and Andrey Malenko, 2013, Means of payment and timing of mergers and acquisitions in a dynamic economy, *Working Paper*.

Grenadier, Steven R., and Neng Wang, 2005, Investment timing, agency, and information, *Journal of Financial Economics* 75, 493–533.

Haile, P., K. Hendricks, and R. Porter, 2010, Recent us offshore oil and gas lease bidding: A progress report, *International Journal of Industrial Organization* 28, 390–396.

Hansen, R.G., 2001, Auctions of companies, *Economic Inquiry* 39, 30–43.

Hansen, Robert G., 1985, Auctions with contingent payments, *American Economic Review* 75(4), 862–865.

Hart, O.D., and J. Moore, 1995, Debt and seniority: An analysis of the role of hard claims, *American Economic Review* 85.

Hellmann, T., 2006, Ipos, acquisitions, and the use of convertible securities in venture capital, *Journal of Financial Economics* 81, 649–679.

Hendricks, K., and R.H. Porter, 1992, Joint bidding in federal ocs auctions, *The American Economic Review* 82, 506–511.

——— , 1996, The timing and incidence of exploratory drilling on offshore wildcat tracts, *The American Economic Review* pp. 388–407.

——— , and G. Tan, 1993, Optimal selling strategies for oil and gas leases with an informed buyer, *The American Economic Review* pp. 234–239.

Kaplan, Steve, and Per Stromberg, 2003, Financial contracting theory meets the real world: an empirical analysis of venture capital contracts, *Review of Economic Studies* 70, 281–315.

Karatzas, I., and S. Shreve, 1988, *Brownian Motion and Stochastic Calculus* (New York: Springer-Verlag).

Klemperer, Paul, 2002, How (not) to run auctions: the european 3g telecom auctions, *European Economic Review* 46, 829–845.

Kogan, S., and J. Morgan, 2010, Securities auctions under moral hazard: An experimental study*, *Review of Finance* 14, 477.

Krishna, V., 2009, *Auction theory* (Academic press).

Laffont, J.J., and J. Tirole, 1987, Auctioning incentive contracts, *The Journal of Political Economy* 95, 921–937.

Landler, Mark, 1997, Fcc offers new options on wireless, *New York Times* September 26, D1.

Maeland, Joril, 2002, Valuation of irreversible investments and agency problems, *Innovation, Organization and Strategy (Trigeorgis, L.), Cambridge University Press, forthcoming.*

——— , 2010, Asymmetric information and irreversible investments: an auction model, *Multinational Finance Journal* 14(3/4), 251–285.

Malmendier, U., M.M. Opp, and F. Saidi, 2011, Cash is king revaluation and the medium of exchange in merger bids, *Working Paper.*

Martin, K.J., 1996, The method of payment in corporate acquisitions, investment opportunities, and management ownership, *The Journal of Finance* 51, 1227–1246.

McAfee, R. Preston, and John McMillan, 1986, Bidding for contracts: a principal-agent analysis, *The RAND Journal of Economics* pp. 326–338.

——— , 1987a, Auctions with entry, *Economics Letters* 23, 343–347.

——— , 1987b, Competition for agency contracts, *The Rand Journal of Economics* pp. 296–307.

McDonald, Robert, and David Siegel, 1986, The value of waiting to invest, *Quarterly Journal of Economics* 101, 707–7288.

Milgrom, Paul, 1985, Auction theory, *Lecture for the Fifth World Congress of the Econometric Society, Mimeo.*

Povel, P., and R. Singh, 2010, Stapled finance, *The Journal of Finance* 65, 927–953.

Protter, P., 2004, *Stochastic Integration and Differential Equations, Second Edition* (New York: Springer-Verlag).

Rappaport, A., and M.L. Sirower, 1999, Stock or cash? the trade-offs for buyers and sellers in mergers and acquisitions., *Harvard business review* 77, 147.

Reece, D.K., 1979, An analysis of alternative bidding systems for leasing offshore oil, *The Bell Journal of Economics* pp. 659–669.

Rhodes-Kropf, Matthew, and S. Viswanathan, 2000, Corporate reorganizations and non-cash auctions, *Journal of Finance* 55(4), 1807–1849.

——— , 2005, Financing auction bids, *RAND Journal of Economics* 36(4), 789–815.

Riley, John G., 1988, Ex post information in auctions, *Review of Economic Studies* 55(3), 409–429.

Robinson, M.S., 1984, Oil lease auctions: Reconciling economic theory with practice, *University of California at Los Angeles, mimeo.*

Rothkopf, M.H., and R. Engelbrecht-Wiggans, 1992, Innovative approaches to competitive mineral leasing, *Resources and Energy* 14, 233–248.

Salant, D.J., 1997, Up in the air: Gte's experience in the mta auction for personal communication services licenses, *Journal of Economics & Management Strategy* 6, 549–572.

Samuelson, W.F., 1985, Competitive bidding with entry costs, *Economics Letters* 17, 53–57.

Samuelson, W., 1987, Auctions with contingent payments: Comment, *The American Economic Review* 77, 740–745.

Skrzypacz, Andrzej, 2013, Auctions with contingent payments-an overview, *International Journal of Industrial Organization*.

Stein, J.C., 2003, Agency, information and corporate investment, *Handbook of the Economics of Finance* 1, 111–165.

Taylor, H., and S. Karlin, 1975, *A first course in stochastic processes* (Academic Press, New York).

Topkis, Donald M., 1978, Minimizing a submodular function on a lattice, *Operations Research* 26(2), 305–21.

Vishwasrao, S., 2007, Royalties vs. fees: How do firms pay for foreign technology?, *International Journal of Industrial Organization* 25, 741–759.

Zheng, Charles Z., 2001, High bids and broke winners, *journal of Economic Theory* 100(1), 129–171.

Appendix

A. Arrow-Debreu Security

Because for sufficiently small dt, $D(P_t; P') = e^{-rdt}\mathbb{E}[D(P_t + dt; P')]$, an application of Itô's formula shows $D(P_t; P')$ satisfies $\frac{1}{2}\sigma^2 P^2 D_{PP} + \mu P D_P - rD = 0$, subject to the boundary conditions $D(P'; P') = 1$ and $D(0; P_0) = 0$. This yields $D(P_t; P') = \mathbb{E}[e^{-r(\tau - t)}] = (\frac{P_t}{P'})^\beta$ where $\tau = \inf\{s \geq t : P_s \geq P'\}$.

B. Proof of Proposition 2.1

Proof. This is a general proof with auction entry fee/cost c, the proposition corresponds to $c = 0$. For $s_1 < s_2$ and $\theta_1 < \theta_2$, condition (R1) implies

$$\ln\left(\frac{V(s_1, \theta_1)V(s_2, \theta_2)}{V(s_1, \theta_2)V(s_2, \theta_1)}\right) = \int_{s_1}^{s_2} \frac{\partial V(s', \theta_2)}{\partial s} ds' - \int_{s_1}^{s_2} \frac{\partial V(s', \theta_1)}{\partial s} ds' < 0 \tag{27}$$

Thus $V(s, \theta)$ is log-submodular, implying it is strictly submodular. Let $Q(s)$ be the probability of winning. Because $s(\theta) \in argmax_s[Q(s)V(s, \theta) - c] = argmax_s \ln(Q(s)V(s, \theta))$, by Topkis (1978), $s(\theta)$ is weakly decreasing in θ. If $s(\theta) < s_H$ were constant on an interval, then the bidder with the lower θ can increase her bid marginally and increase her probability of winning (thus her payoff) by a discrete amount. Therefore $s(\theta)$ must be decreasing in type for the types bidding less than s_H. From this we get $Q(s(\theta)) = (1 - F(\theta))^{N-1}$ for these types. s is also continuous in θ, lest a type right below a discontinuity could lower her bid marginally without affecting the chance of winning.

Next, by direct revelation, $\theta \in \text{argmax}_{\theta' \in} Q(s(\theta'))V(s(\theta'), \theta)$. For any $\theta' < \theta$,

$$Q(s(\theta))V(s(\theta), \theta) \geq Q(s(\theta'))V(s(\theta'), \theta) = Q(s(\theta'))[V(s(\theta), \theta) + V_1(s^*, \theta)[s(\theta') - s(\theta)]]$$

for some s^* between $s(\theta')$ and $s(\theta)$. Since $V_1 < 0$, the above expression can be written as

$$\frac{Q(s(\theta')) - Q(s(\theta))}{s(\theta') - s(\theta)} \frac{V(s(\theta), \theta)}{-Q(s(\theta'))V_1(s^*, \theta)} \leq 1$$

i.e.

$$\frac{Q(s(\theta')) - Q(s(\theta))}{\theta' - \theta} \frac{V(s(\theta), \theta)}{-Q(s(\theta'))V_1(s^*, \theta)} \geq \frac{s(\theta') - s(\theta)}{\theta' - \theta}$$

Similarly, exchanging θ and θ', for some s^{**} between $s(\theta)$ and $s(\theta')$,

$$\frac{Q(s(\theta')) - Q(s(\theta))}{\theta' - \theta} \frac{V(s(\theta'), \theta')}{-Q(s(\theta))V_1(s^{**}, \theta')} \leq \frac{s(\theta') - s(\theta)}{\theta' - \theta}$$

Taking the limit we get

$$s'(\theta) = \frac{(N-1)f(\theta)}{1 - F(\theta)} \frac{V(s(\theta), \theta)}{V_1(s(\theta), \theta)} \tag{28}$$

which characterizes the bidding strategy.

For the boundary condition, let $\hat{\theta}$ be the cut-off type for IR of participation in the auction, then $S(s(\hat{\theta}), \cdot) = 0$ because if $\hat{\theta}$ is the smallest realized θ, then all other potential bidders are not participating and $\hat{\theta}$ takes the object without bidding any security, and if it is not the smallest θ, it doesn't win the object no matter how much security it is bidding as it is the worst type that is participating. The IR constraint dictates that $Q(s(\hat{\theta}))V(s(\hat{\theta}), \hat{\theta}) \geq c$. However, if $Q(s(\hat{\theta}))V(s(\hat{\theta}), \hat{\theta}) > c$, then the type right above $\hat{\theta}$ can increase s marginally from zero and participate, and earn positive expected profit. Thus $Q(s(\hat{\theta}))V(s(\hat{\theta}), \hat{\theta}) = c$.

Let $G(\theta) = [1 - F(\theta)]^{N-1}(W(\theta) - X)$ denote the expected payoff from participating. Because of the monotonicity of G over the support of θ, either there is no participants at all or $G(\underline{\theta}) > c$ and because the worst type would not want to participate, $G(\overline{\theta}) < 0$. In the latter case, there always exists a $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ that defines the cut-off type, which is the same as in the cash-bid auctions.

This establishes the uniqueness, we verify existence by showing the sufficiency of the bidder's first order condition, specifically, the quasiconcavity of $\ln(Q(s)V(s, \theta))$. For any $s' \in (s(0), s(\theta))$, $\exists \theta' \in (0, \theta)$ such that $s(\theta') = s'$. Then (R1) implies $\frac{\partial}{\partial s} \ln[Q(s')V(s', \theta)] > \frac{\partial}{\partial s} \ln[Q(s')V(s', \theta')] = 0$. A similar argument leads to $\frac{\partial}{\partial s} \ln[Q(s')V(s', \theta)] < 0$ for $s' \in (s(\theta), s(\underline{\theta}))$. Therefore for every θ, there exists a s that maximizes $Q(s)V(s, \theta) - c$. □

C. Proof of Proposition 2.2

Proof. Again this is a general proof with auction entry fee/cost c. An argument similar to that for V being non-increasing in s leads to V being decreasing in θ. Thus a bidder participating in equilibrium is always better off increasing the bid until $V(s, \theta) = 0$. To see this, suppose the LHS is bigger and we increase s. In the states of the world where we could have won the auction, we are still paying the same second highest bid, so our profit is not reduced. In the states of the world where we would not have won the auction but win the auction after this increase, we are still making non-negative profit as the value we get is $V(s',\theta) \geq V(s,\theta) > 0$. Similarly, we want to decrease s when $V(s,\theta) < 0$. Since by varying s, we can reach any value in $[0, P_\tau]$, we can always increase it till $V(s,\theta) = 0$. Again, the cut-off type wins only if other types are not participating, in which case she pays no contingent security to the seller. This leads to the cut-off being the same as in the case of cash-bid auctions and FPAs with security bids, i.e., $\hat{\theta}$. Total differentiation of $V(s,\theta)$ leads to $s(\theta)$ being decreasing in θ. □

D. Proof for Proposition 2.3

Proof. When P_0 increases or X decreases, $\hat{\theta}$ weakly increases, thus potentially a positive measure of originally non-participating bidders are bidding greater s. Given the bidding strategy in SPAs and the fact V is non-increasing in s, decreasing in X and increasing in P_0, for each type that is bidding increasing P_0 or decreasing X would result in a bigger s. In FPAs, let $\tilde{s}(\theta)$ denote the bidding strategy after P_0 increases or X decreases. Then $\tilde{s}(\theta) \geq s(\theta)$ at lesat for the original cutoff type $\hat{\theta}_{old}$. If $\tilde{s}(\theta) = s(\theta)$ for any $\theta \in [\underline{\theta}, \hat{\theta}_{old}]$, $\tilde{s}'(\theta) > s'(\theta)$, thus $\tilde{s}(\theta)$ stays above $s(\theta)$ for a positive measure of types. Hence overall we have a positive measure of types bidding bigger s, and all types bidding weakly bigger s. For the same reason, the result holds when N increases in FPAs. □

E. Proof for Proposition 3.1

I first derive an optimal strategy among all threshold strategies and then verify it is indeed optimal in the space of all strategies. Note the threshold cannot be less than P_0 because the exercise payoff to the bidder is increasing in P and the bidder can always do better by invest right away at P_0. For a strategy with threshold $P \geq P_0$, differentiate the payoff $D(P_0; P)((1-\alpha)P-\theta)$ w.r.t. P, and the F.O.C. and S.O.C. give the optimal threshold $P^{equity}(\theta)$.

To verify the optimality of the strategy, let $x_t = e^{-rt}\hat{W}(P_t)$, where $\hat{W}(P_t) = D(P_t; \hat{P}(\theta))[(1-\alpha)\hat{P} - \theta]\}$ and $\hat{P}(\theta) = \max\{P_t, \frac{\beta\theta}{(\beta-1)(1-\alpha)}\}$. Using an extended version of Itô's formula (as, for example, in Karatzas and Shreve (1988), page 219), $dx_t = e^{-rt}[\mathcal{D}\hat{W}(P_t) - r\hat{W}(P_t)]dt + e^{-rt}\hat{W}_P(P_t)\sigma P_t dB_t$, where $\mathcal{D}\hat{W}(P) = \hat{W}_P(P)\mu P + \frac{1}{2}\hat{W}_{PP}(P)\sigma^2 P^2$ except at $P = \frac{\beta\theta}{(\beta-1)(1-\alpha)}$, where we may replace $\hat{W}_{PP}(P^*)$ with zero. $\hat{W}_P$ is bounded as seen by direct computation, thus by Proposition 5B in Duffie (2009) (also found in Protter (2004)), the last term in dx_t

is a martingale under the current measure. For $P \leq \frac{\beta\theta}{(\beta-1)(1-\alpha)}$, $\mathcal{D}\hat{W}(P) - r\hat{W}(P) = 0$ by the definition of β in (4). For $P > \frac{\beta\theta}{(\beta-1)(1-\alpha)}$, $\mathcal{D}\hat{W}(P) - r\hat{W}(P) = r\theta - (r-\mu)(1-\alpha)P < r\theta - (r-\mu)(1-\alpha)\hat{P}(\theta) = \frac{\theta(\mu\beta - r)}{\beta-1} = -\frac{1}{2}\theta\beta\sigma^2 < 0$. The drift of x is thus never positive, implying for any stopping time τ, $\hat{W}(P_0) \geq \mathbb{E}[e^{-r\tau}\hat{W}(P_\tau)] \geq \mathbb{E}[e^{-r\tau}((1-\alpha)P_\tau - \theta)]$ with equality holding for the first-hitting time of $P^{equity}(\theta)$. This implies the optimality of the threshold strategy with trigger $P^{equity}(\theta)$. This verification scheme applies generally to subsequent proofs of optimality of strategies in this paper, with minor modification to the argument for the local martingale to be a martingle.

F. Proof of Lemma 3.2

Proof. If a bidder of type θ bids a strike less than $X + \theta$, with positive probability she wins with a required strike $k < X + \theta$ in both FPA and SPA, and fails to break even. So she is better of bidding $k \geq X + \theta$. If she bids a strike greater than $P^*(\theta)$, the required strike conditional on winning in either FPA or SPA satisfies $k > P^*(\theta)$ and the bidder always invests with the threshold $P^*(\theta)$ and the call is never exercised. But she could reduce lower k in the bid to increase the chance of winning. Hence $k \in [X + \theta, P^*(\theta)]$.

Assuming a threshold strategy, maximizing the payoff $D(P_0; P)[P - max\{P-k, 0\} - \theta] - X$ w.r.t. P gives the expression for $P^{call}(\theta)$. Next, we verify the optimality of the threshold strategy. Again, let $x_t = e^{-rt}\hat{W}(P_t)$, where $\hat{W}$ denotes the option value by following the threshold strategy with trigger $P^{call}(\theta) = \max\{P_t, k\}$. Given $k \in [X + \theta, P^*(\theta)]$, if $P_t \leq k$, the drift in dx is zero by the definition of β in (4). If $P_t > k$, the drift is $-r(k-\theta) < 0$. The drift of dx is thus never positive. Thus an argument similar to the verification proof in equity auctions proves the proposed threshold strategy is indeed optimal. □

G. Proof of Lemma 4.1

Proof. The IC constraint can be written as $\theta_i \in \text{argmax}_{\tilde{\theta}_i \in [\underline{\theta}, \bar{\theta}]} U(\theta_i, \tilde{\theta}_i, \tau^*(\tilde{\theta}_i, S))$. As $U(\theta_i, \tilde{\theta}_i, \tau^*)$ is not necessarily differentiable in $\tilde{\theta}_i$, rewrite this as $\tilde{\theta}_i \in \text{argmax}_{\theta_i \in [\underline{\theta}, \bar{\theta}]}[U(\theta_i, \tilde{\theta}_i, \tau^*(\tilde{\theta}_i, S)) - U(\theta_i)]$. By bounded convergence theorem, $U(\theta_i, \tilde{\theta}_i, \tau^*(\tilde{\theta}_i, S))$ is differentiable in θ_i, with

$$U'(\tilde{\theta}_i) = \frac{\partial U(\theta_i, \tilde{\theta}_i, \tau^*(\tilde{\theta}_i, S))}{\partial \theta_i}|_{\theta_i = \tilde{\theta}_i} = \mathbb{E}_{\theta_{-i}}\left[Q(\tilde{\theta}_i, \theta_{-i})\mathbb{E}_P[-e^{-r\tau_i}]\right] \tag{29}$$

which is uniformly bounded by 1, implying $U(\theta_i)$ being Lipschitz continuous because it is the upper envelope of a family of Lipschitz continuous functions $U(\theta_i, \tilde{\theta}_i, \tau^*(\tilde{\theta}_i, S))$ indexed by $\tilde{\theta}_i$ with the same Lipschitz constant (Proposition 6.3 in Choquet (1966)). $U(\theta_i)$ is thus absolute continuous, and differentiable almost everywhere (Corollary 6.3.7, Cohn (1994)). Writing it in the integral form concludes the proof.

□

H. Proof of Proposition 4.1 and Section 5.2

Proof. The derivation is for the general case in Section 5.2. Let the investment-independent payment from θ_i be $L(\theta_i, \theta_{-i})$. Let L_i denote bidder i's liquidity constraint and L_0 the auctioneer's. Proposition 4.1 corresponds to $L_i = L_0 = 0$. The ex-ante social welfare is

$$N\mathbb{E}_{\vec{\theta}}[Q(\theta_i, \theta_{-i}) \left(\mathbb{E}_P[e^{-r\tau^*(\theta_i,\theta_{-i},S)}(P_{\tau^*(\theta_i,\theta_{-i},S)} - \theta_i)] - X\right)] \tag{30}$$

and the seller's *ex ante* revenue is the social welfare less the agents' ex-ante utilities.

$$\begin{aligned} R =& N\mathbb{E}_{\vec{\theta}}[Q(\theta_i, \theta_{-i})(\mathbb{E}_P[e^{-r\tau^*(\theta_i,\theta_{-i},S)}(P_{\tau^*(\theta_i,\theta_{-i},S)} - \theta_i)] - X)] - N\mathbb{E}_{\vec{\theta}}[U(\theta_i)] \\ =& N\mathbb{E}_{\vec{\theta}}[Q(\theta_i, \theta_{-i})(\mathbb{E}_P[e^{-r\tau^*(\theta_i,\theta_{-i},S)}(P_{\tau^*(\theta_i,\theta_{-i},S)} - \theta_i)] - X)] \\ &- N\mathbb{E}_{\vec{\theta}}[Q(\theta_i, \theta_{-i})\mathbb{E}_P[e^{-r\tau^*(\theta_i,\theta_{-i},S)}F(\theta_i)/f(\theta_i)]] - NU(\overline{\theta}) \\ =& N\mathbb{E}_{\vec{\theta}}[Q(\theta_i, \theta_{-i})(\mathbb{E}_P[e^{-r\tau^*(\theta_i,\theta_{-i},S)}(P_{\tau^*(\theta_i,\theta_{-i},S)} - \theta_i - F(\theta_i)/f(\theta_i))] - X)] - NU(\overline{\theta}) \end{aligned} \tag{31}$$

where the second lines comes from taking expectations of the winning bidder's type and integrating (14) by parts.

To maximize revenue, for every realization of the types and any allocation rule, winner θ_i is required to invest when P first hits $P_{\tau^*}(\theta_i) = \max\{P_0, (\theta_i + F(\theta_i)/f(\theta_i))\beta/(\beta - 1)\}$ by the same verification argument in Appendix E with the investment cost replaced by $\theta_i + F(\theta_i)/f(\theta_i)$. Moreover, $U(\overline{\theta}) = 0$ and the project is only allocated to types that contribute positively to the revenue, i.e., the cut-off type solves $\mathbb{E}_P[D(P_0; P_{\tau^*}(\hat{\theta}))(P_{\tau^*}(\hat{\theta}) - \hat{\theta} - F(\hat{\theta})/f(\hat{\theta}))] = X$. The assumption on the inverse hazard rate function leads to the unique cutoff type and the allocation rule given in the proposition.

That $U(\theta_i)$ is decreasing in θ_i implies any mechanism satisfying the above meets IR of all types. Suppose $\theta_i < \tilde{\theta}_i$, Lemma 4.1 leads to

$$\begin{aligned}
&U(\theta_i, \tilde{\theta}_i, \tau^*(\theta_i, z_i(\tilde{\theta}_i, \theta_{-i}, \cdot))) = U(\tilde{\theta}_i) - \int_{\theta_i}^{\tilde{\theta}_i} \frac{\partial}{\partial \theta_i} U(\theta, \tilde{\theta}_i, \tau^*(\theta, z_i(\tilde{\theta}_i, \theta_{-i}, \cdot)))d\theta \\
&\leq U(\tilde{\theta}_i) - \int_{\theta_i}^{\tilde{\theta}_i} \frac{\partial}{\partial \theta_i} U(\theta, \theta, \tau^*(\theta, z_i(\theta, \theta_{-i}, \cdot)))d\theta = U(\theta_i) \qquad (32)
\end{aligned}$$

where the inequality follows from the fact that reporting a higher investment cost leads to a lower probability of winning and is required to invest later. Similarly, $U(\theta_i, \tilde{\theta}_i, \tau^*(\theta_i, z_i(\tilde{\theta}_i, \theta_{-i}, \cdot))) \leq U(\theta_i)$ for $\theta_i > \tilde{\theta}_i$. Thus incentive compatibility holds.

Finally, investment-independent and contingent payments are such that Lemma 4.1 holds. For every investment-independent payment schedule $L(\cdot, \cdot)$ satisfying the

(1) ***Budget Constraints*** $L(\theta_i, \theta_{-i}) \in [-L_0, L_i]$ for all θ_i, θ_{-i}.

(2) ***Limited Liability*** of $S(\tilde{\theta}_i, \theta_{-i}, \{P_t, t \leq \tau\}, \tau^*)$: $L(\theta_i, \theta_{-i})$ satisfies $L(\theta_i, \theta_{-i}) \geq -X$ and

$$\frac{\mathbb{E}[Q(\theta_i, \theta_{-i})L(\theta_i, \theta_{-i})]}{[1 - F(\theta_i)]^{N-1}} \leq D(P_0; P_{\tau^*}(\theta_i))[P_{\tau^*}(\theta_i) - \theta_i] - X - \int_{\theta_i}^{\hat{\theta}} \left[\frac{1 - F(\theta)}{1 - F(\theta_i)}\right]^{N-1} D(P_0; P_{\tau^*}(\theta))d\theta,$$

there exists investment-contingent payment schedule(s) characterized by the ***Balance Equation***

$$\begin{aligned}
&\int_{\theta_i}^{\hat{\theta}} [1 - F(\theta)]^{N-1} D(P_0; P_{\tau^*}(\theta))d\theta \qquad (33) \\
=&\mathbb{E}_{\theta_{-i}} \left(Q(\theta_i, \theta_{-i}) \left[D(P_0; P_{\tau^*}(\theta_i))[P_{\tau^*}(\theta_i) - \theta_i - S(\tilde{\theta}_i, \theta_{-i})] - X - L(\theta_i, \theta_{-i}) \right] \right).
\end{aligned}$$

Any such pair constitute an optimal design. Conversely, for every investment-contingent payment schedule $S(\cdot, \cdot)$, if an investment-independent schedule satisfies the above balance equation and is within the budget constraint, it is an optimal design.

One special case is when $S(\theta_i, \theta_{-i}) = \frac{F(\theta_i)}{f(\theta_i)}$ and the L schedule satisfies the balance equation (The case in Board (2007b) where there is no budget constraints is included), then one does not need to contract on P. To implement it, use the contingent payment schedule and let agents bid upfront cash in a FPA. A monotone symmetric bidding equilibrium allows backing out the types, rendering this arrangement feasible. Let $b_{FPA}(\theta_i)$ be a decreasing strategy, then in a first price auction, Prop. 4.1 implies,

$$\int_{\theta_i}^{\hat{\theta}}[1-F(\theta)]^{N-1}\mathbb{E}_P[e^{-r\tau_\theta^*}]d\theta+c = [1-F(\theta_i)]^{N-1}\mathbb{E}_P\left[e^{-r\tau_{\theta_i}^*}\left(P_{\tau_{\theta_i}^*} - \theta_i - F(\theta_i)/f(\theta_i)\right) - X - b_{FPA}(\theta_i)\right]$$

It can be verified that non-decreasing $F(\theta)/f(\theta)$ leads to $b_{FPA}(\theta_i)$ being decreasing and the cut-off type of participation is as described.

Another special case is when both seller and bidders are completely liquidity constrained, which leads to the results stated in Prop. 4.1.

Notice the type with least investment cost is also the type with the highest post-auction utility because the post-auction utility is simply $U(\theta_i)/[1-F(\theta_i)]^{N-1}$ of which the derivative w.r.t. θ_i is negative. Thus ex post utility decreases with investment cost.

□

I. Proof of Proposition 4.2

Proof. This follows directly from maximizing the social welfare. The ***Budget Constraint, Limited Liability and Balance Equation*** need to be satisfied now with $P_{\tau^*}(\theta_i) = \frac{\beta}{\beta-1}\theta_i$. Monotone property of the expected utility is apparent. Least θ corresponds to the highest ex post utility. Now the seller picks the cut-off so that the project is only allocated to bidders who contribute positively to the welfare, which coincides with the cut-off in the cash-bid auctions. □

J. Proof for Prop 4.3

Proof. To see this, the auctioneer's expected objective at P_{00} when she plans to hold the auction when P_0 can be written as

$$D(P_{00}; P_0)\int_{\underline{\theta}}^{\hat{\theta}} Nf(\theta)[1-F(\theta)]^{N-1}[W(P_0; z(\theta)) - X]d\theta \tag{34}$$

where $z(\theta) = \theta$ in welfare-maximizing auctioneer and $z(\theta) = \theta + F(\theta)/f(\theta)$ for the revenue-maximizing auctioneer. The derivative w.r.t. P_0 is

$$\frac{P_{00}^{\beta}}{P_0^{\beta+1}}\int_{\underline{\theta}}^{\hat{\theta}} Nf(\theta)[1-F(\theta)]^{N-1}[\beta X - (\beta-1)P^*(z(\theta)) + \beta z(\theta)]d\theta \tag{35}$$

When $P_0 < \frac{\beta}{\beta-1}\underline{\theta}$, $-(\beta-1)P^*(z(\theta)) + \beta z(\theta) = 0 \quad \forall \quad \theta \in [\underline{\theta}, \hat{\theta}]$ because $P^*(z) = \frac{\beta}{\beta-1}z$. Thus the derivative is positive, making delaying slightly a positive deviation. □

K. Proof for Prop 4.4

Proof. Just a sketch: First, $\exists \tilde{\theta} \le \overline{\theta}$ s.t. $\int_{\tilde{\theta}}^{\overline{\theta}} Nf(\theta)[1-F(\tilde{\theta})]^{N-1}\beta X < A$ where A is a constant independent of the model parameters. Now for $P_0 \ge \frac{\beta}{\beta-1}\tilde{\theta}$, the derivative is bounded above by

$$\frac{P_{00}^{\beta}}{P_0^{\beta+1}}\left[\int_{\underline{\theta}}^{\tilde{\theta}} Nf(\theta)[1-F(\theta)]^{N-1}[\beta(X+z(\theta)) - (\beta-1)P_0]d\theta + A\right] \tag{36}$$

which is strictly decreasing without lower bound in P_0. Thus the derivative w.r.t. P_0 becomes negative and stays negative for $P_0 \ge \overline{P}$ where $\overline{P}$ is finite. This implies that the objective is decreasing for $P_0 \ge \tilde{P}$. Since the expected objective varies continuously in the bounded interval $[P^*(\theta), \overline{P}]$, thus a maximum exists. The corresponding P_0 would be the threshold. This only shows that the timing strategy is optimal among all threshold strategies, a verification argument similar to the ones in Appendix E and F gives optimality among all timing strategies. □

L. Proof for Prop 4.5

Proof. In the region $P_0 \ge \frac{\beta}{\beta-1}\underline{\theta}$, $\hat{\theta} = \overline{\theta}$. The derivative is an integration over $[\underline{\theta}, \overline{\theta}]$ of an expression that is weakly decreasing for all P_0 in the region and strictly decreasing over some region of P_0, thus the derivative changes sign only once, which gives an unique solution to the F.O.C.. The objective is quasi-concave by argument in the previous proof. Thus the solution to the F.O.C. gives an optimal threshold for timing. In this region, the integrand is also increasing in z. Thus when the derivative is zero for $z(\theta) = \theta$, it is positive for $z(\theta) = \theta + F(\theta)/f(\theta)$, implying the threshold trigger is higher for the optimal mechanism. The auction is held later in an optimal mechanism than in an efficient mechanism. □

M. Informal Auctions

9.0.1 Proof for Lemma 5.1

Proof. We first show that $R(S^i) = R_{\theta_i}(S^i)$. This is obviously true if only one type uses S^i. If more than one type use this bid, either this is true or one of the types θ_1 has $R_{\theta_1}(S^i) \ne R(S^i)$.

Then $\exists$ type θ_2 (potentially $= \theta_1$) s.t. $R(S^i) < R_{\theta_2}(S^i)$. Consider the deviation for bidder 2 to bid $\hat{S}$ such that $\hat{S}(P^*(\theta_2)) = [R(S^i)/D(P_0; P^*(\theta_2))]\mathbf{1}_{\{P=P^*(\theta_2)\}} + P\mathbf{1}_{\{P\neq P^*(\theta_2)\}}$. Since bidder i only gets paid when investing at $P^*(\theta_2)$, she invests when cash flow first hits this level, i.e., the investment strategy is τ_2^*. This is profitable because she creates weakly greater social surplus, pays less, and has the same marginal probability of winning. Thus $R(S_{\theta_i}) = R_{\theta_i}(S^i)$ always.

Now suppose $\tau^i \neq \tau_i^*$, then there is a profitable deviation. Consider deviating to $\hat{S}(P) = [R(S^i)/D(P_0; P^*(\theta_i))]\mathbf{1}_{\{P=P^*(\theta_i)\}} + P\mathbf{1}_{\{P\neq P^*(\theta_i)\}}$. Since $R(S^i) \leq \mathbb{E}[e^{-r\tau^i}(P_{\tau^i} - \theta_i)] < \mathbb{E}[e^{-r\tau_i^*}(P_{\tau_i^*} - \theta_i)] < D(P_0; P^*(\theta_i))P^*(\theta_i)$, $\hat{S}(P) \leq P$ $\forall P$ and this bid is valid. The payoff from deviation $\mathbb{E}[e^{-r\tau_i^*}(P_{\tau_i^*} - \theta_i)] - R(\hat{S})$ dominates the original payoff $\mathbb{E}[e^{-r\tau^i}(P_{\tau^i} - \theta_i) - e^{-r\tau^i}S_{\theta_i}(s, P_{\tau^i})]$. The claim follows.

□

9.0.2 Proof for Lemma 5.2

Proof. Suppose a set Θ_p of types pool to bid S. From Lemma 5.1, a type θ in expectation pays $D(P_0; P^*(\theta))S(P^*(\theta))$. Let $k = \text{argmax}_{k'\in\Theta_p} R_{\theta_{k'}}(S)$ where $R_{\theta_{k'}}(S) = D(P_0; P^*(\theta_{k'}))S(P^*(\theta_{k'}))$. Then $R(S) \leq D(P_0; P^*(\theta_k))S(P^*(\theta_k))$. If the inequality is strict, type k can profitably deviate to $\hat{S}(P) = [R/D(P_0; P^*(\theta_k))]\mathbf{1}_{\{P=P^*(\theta_k)\}} + P\mathbf{1}_{\{P\neq P^*(\theta_k)\}}$. Otherwise, $R_{\theta_i}(S) = R_{\theta_j}(S) = R(S)$, and there is still a contradiction as argued next.

We first argue that Θ_p contains a positive measure of types. Take $\theta_i < \theta_j$ both in Θ_p, for any $\theta_n \in (\theta_i, \theta_j) \bigcap \Theta_p^c$, call her bid $\tilde{S}$. Let Q and $\tilde{Q}$ be the probability of winning when bidding S and $\tilde{S}$. Since θ_i does not want to deviate to $\hat{S}_i(P) = \min\{R(S')/D(P_0; P^*(\theta_i)), P\}\mathbf{1}_{\{P=P^*(\theta_i)\}} + P\mathbf{1}_{\{P\neq P^*(\theta_i)\}}$, $Q[W(\theta_i) - R(S) - X] \geq \tilde{Q}[W(\theta_i) - R(S') - X]$. Similarly, $Q[W(\theta_j) - R(S) - X] \geq \tilde{Q}[W(\theta_j) - R(S') - X]$. As $\theta_i \neq \theta_j$, the equality signs cannot hold simultaneously. Thus for $\theta_n \in (\theta_i, \theta_j)$, $Q[W(\theta_n) - R(S) - X] > \tilde{Q}[W(\theta_n) - R(S') - X]$. This means θ_n can profitably deviate to $\hat{S} = \min\{R(S)/D(P_0; P^*(\theta_n)), P\}\mathbf{1}_{\{P=P^*(\theta_n)\}} + P\mathbf{1}_{\{P\neq P^*(\theta_n)\}}$. Therefore, $[\theta_i, \theta_j] \in \Theta_p$.

Now we show there is a contraction. As $W(P_0; \theta_i) - X - R_{\theta_i}(S) > W(P_0; \theta_j) - X - R_{\theta_j}(S) \geq 0$. The type θ_i can deviate profitably to a $\hat{S}(P) = [(\epsilon + R)/D(P_0; P^*(\theta_i)]\mathbf{1}_{\{P=P^*(\theta_i)\}} + P\mathbf{1}_{\{P\neq P^*(\theta_i)\}}$ which reduces her payoff by ϵ upon winning but increases her marginal chance of winning by a discrete amount (because she separates from a positive measure of types). Since there is always a profitable deviation by a type in the pooling set, the claim follows. □

9.0.3 Proof for Prop. 5.1

Proof. Consider the bidding strategy from a first-price cash auction. The valuations for the bids are simply the cash amounts themselves. We only need to show that there exists a belief that supports an equilibrium with this bidding strategy in the informal auction. First, there would not be any deviation to another cash amount since the bidding strategy comes from the equilibrium in FPA cash auction. Next, for beliefs such that upon seeing an out-of-equilibrium bid S^i, the auctioneer believes it comes from $\tilde{\theta}_i = \text{argmin}_{\theta \in [\underline{\theta}, \overline{\theta}]} R_\theta(S^i)$ and gives it a valuation $\tilde{R}$. If bidder i finds this deviation attractive (yielding an expected payoff more than the original cash amount she is paying), then she also finds deviating to paying a cash of $\tilde{R}$ weakly more attractive, contradicting the fact that no deviation to another cash amount is profitable. Thus when cash is available, the equilibrium from a first-price cash auction is an equilibrium in the informal auction. Suppose bidder i pays a cash z_i in the first-price cash auction, even if cash is not available, the argument still applies to the following bidding strategy with bidder i bids $\hat{S}^i(P) = [z_i/D(P_0; P^*(\theta_i)]\mathbf{1}_{\{P=P^*(\theta_i)\}} + P\mathbf{1}_{\{P \neq P^*(\theta_i)\}}$. These are cash-like bids because their valuations are independent of the auctioneer's belief on the bidders' types.

Next I show any equilibrium in the informal auction has the same expected payoffs as the cash auction. Lemma 5.2 tells us that in equilibrium there is no pooling. Thus the auctioneer forms the correct beliefs about types. Bidder i's bid S^i can be replaced by a cash-like bid (or cash bid if available) whose value is $\tilde{z}_i = \mathbb{E}[e^{-r\tau_i} S_{\theta_i}(P_{\tau_i})]$. Note $\tau_i = \tau_i^*$ from Lemma 5.1. This would not change the marginally probability of winning, neither does it change the payoff upon winning. Since $\tilde{z}_i$s solve the same maximization problem faced by bidders in a first-price cash auction, $\tilde{z}_i = z_i$. Thus in terms of expected payoff, almost every bid is a cash-like bid. □

Delays when Seller Initiates Informal Auctions

Proof. Write $G(\theta) = (N-1)F(\theta)/(1-F(\theta))$, then it is increasing in θ and $\int_{\underline{\theta}}^{\overline{\theta}} N f(\theta)[1-F(\theta)]^{N-1} d\theta = 1$. As $H(\theta) \equiv \beta X - (\beta - 1)P^*(z(\theta)) + \beta z(\theta)$ is weakly increasing in θ, and strictly increasing in θ over some region when the F.O.C. in an efficient mechanism holds, at the efficient P_0, $G(\theta)$ and $H(\theta)$ are positively correlated. Thus the derivative of the objective has the same sign as $\mathbb{E}[G(\theta_{(1)})H(\theta_{(1)})] > \mathbb{E}[G(\theta_{(1)})]\mathbb{E}[H(\theta_{(1)})] = 0$, thus the auctioneer prefers to wait longer in timing a second-price cash auction than in an efficient mechanism. □

Proof for Prop. 5.2

[Main steps]

Proof. Conjecture that in the equilibrium, bidder θ initiates the auction with threshold $P_I(\theta)$ such that $P_I(\theta)$ is increasing. And suppose the seller follows a threshold strategy to start the auction at P_S. If bidder θ approaches at P, then the expected payoff is

$$\left(\frac{P_0}{P}\right)^\beta \int_{P_I^{-1}(P)}^{\overline{\theta}} [W(P,\theta) - W(P,\theta')]^+ f(\theta')[1-F(\theta')]^{N-2}(N-1)d\theta'$$
$$+ \int_{\underline{\theta}}^{P_I^{-1}(P)} \left(\frac{P_0}{P_I(\theta')}\right)^\beta [W(P_I(\theta'),\theta) - W(P_I(\theta'),\theta')]^+ f(\theta')[1-F(\theta')]^{N-2}(N-1)d\theta' \quad (37)$$

if $P \leq P_S$; otherwise, it is

$$\left(\frac{P_0}{P_S}\right)^\beta \int_{P_I^{-1}(P_S)}^{\overline{\theta}} [W(P_S,\theta) - W(P_S,\theta')]^+ f(\theta')[1-F(\theta')]^{N-2}(N-1)d\theta'$$
$$+ \int_{\underline{\theta}}^{P_I^{-1}(P_S)} \left(\frac{P_0}{P_I(\theta')}\right)^\beta [W(P_I(\theta'),\theta) - W(P_I(\theta'),\theta')]^+ f(\theta')[1-F(\theta')]^{N-2}(N-1)d\theta' \quad (38)$$

Either $P_I(\theta) > P_S$, or the derivative w.r.t. P is negative when $P > P^*(X+\theta)$, leading to equilibrium $P_I(\theta) \leq P^*(\theta + X)$. Similarly, $P_I(\theta) > P^*(\theta)$. Now for the seller, if she uses threhold P, the expected payoff is,

$$\left(\frac{P_0}{P}\right)^\beta \int_{P_I^{-1}(P)}^{\overline{\theta}} W(P,\theta')^+ f(\theta')F(\theta')[1-F(\theta')]^{N-2}N(N-1)d\theta'$$
$$+ \int_{\underline{\theta}}^{P_I^{-1}(P)} \left(\frac{P_0}{P_I(\theta')}\right)^\beta W(P_I(\theta'),\theta')^+ f(\theta')F(\theta')[1-F(\theta')]^{N-2}N(N-1)d\theta' \quad (39)$$

It cannot be $P > P^*(P_I^{-1}(P) + X)$ in equilibrium lest type $P_I^{-1}(P)$ would initiate strictly before the seller. Thus in the first integrand, $P < P^*(\theta' + X)$ over positive measure because $P^*(\cdot)$ is increasing. The derivative of the whole expression is positive at P for $P_S < P_I(\overline{\theta})$. The seller would never initiate the auction.

Now the bidder's problem is reduced to expression (37). Setting the derivative to zero, and the solution need to match the equilibrium $P_I(\theta)$, we get $P_I(\theta)$ solves

$$\int_{\theta}^{\overline{\theta}} \frac{d}{dP}\frac{W(P,\theta)-W(P,\theta')}{P^{\beta}}|_{P=P_I}(N-1)f(\theta')[1-F(\theta')]^{N-2}d\theta' = 0 \qquad (40)$$

The integrand is weakly monotone path-by-path. LHS is continous and is non-negative at $P_I = P^*(\theta)$ and non-positive at $P^*(\theta+X)$, thus a solution exists. Optimality of FOC comes from the concavity of $[W(P,\theta) - W(P,\theta')]/P^{\beta}$ path-by-path. The monotonicity of $P_I(\theta)$ can be verified by implicitly differentiating the FOC and looking at second derivatives and cross-partials.

An argument can be made why the initiation strategy for the bidders have to be monotone, in order to show uniqueness of the equilibrium. Optimality verification is more involved but would follow similar arguments as in Appendix E. □

N. Proof for Proposition 6.1

Proof. First, moral hazard is absent in cash auctions as the winning bidder is the sole owner of the investment option. Given a bidder of type θ optimally invest at $P^*(\theta)$, the valuation is $W(P_0;\theta) - X$. In cash SPAs, they bid up to their valuation, i.e. $b = \max\{0, W(\theta) - X\}$. Since the valuation is monotone in θ, there is a cut-off type for participation $\hat{\theta}$ which solves $c = (1-F(\theta))^{N-1}[W(\theta) - X]$. In cash FPAs, a participating bidder bids $b(\theta_i) = \mathbb{E}[(W(\theta') - X)|\theta' \geq \theta_i]$, where θ' is the smallest type among the remaining participants. Again, the incentives to participate is monotone in θ, there is a cut-off type $\tilde{\theta}$. Since it is an indifference point, $\tilde{\theta}$ wins only if all other types are not participating, in which case she pays zero security and breaks even. Thus $\tilde{\theta} = \hat{\theta}$. Since the allocation, the cut-off type are identical, the welfares under FPA and SPA with cash bids are the same. An application of Fubini's Theorem establishes revenue equivalence.

Security-bid auctions with entry fee c is already solved in Appendix B and C. If c is an entry cost instead of entry fee, the welfares are modified by subtracting $cNF(\hat{\theta})$. □

O. Proof of Proposition 6.2

Proof. We first prove a useful result that $\max_{\tau} \mathbb{E}[e^{-r\tau}(P_{\tau} - \theta - S(s(\hat{\theta}), P_{\tau}))] = \max_{\tau} \mathbb{E}[e^{-r\tau}(P_{\tau} - \theta)]$ for $\theta \leq \hat{\theta}$. Notice the LHS≤RHS in general, thus we only need to show that there is an optimal strategy that achieves this equality. Take $P_{\tau} = \frac{\beta}{\beta-1}\theta \leq \frac{\beta}{\beta-1}\hat{\theta}$, since $S(s(\hat{\theta}), P)$ is positive and non-decreasing in P, and is zero at $P = \frac{\beta}{\beta-1}\hat{\theta}$, we deduce $S(s(\hat{\theta}), P_\tau) = 0$, and indeed P_τ is the optimal trigger for investment.

With this we take the expected revenue in SPA, and use $\tau(\theta_1, \theta_2)$ to denote the optimal stopping time for type $\theta_{(1)}$ when the second highest type is θ_2,

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{\theta_{(2)}<\hat{\theta}\}} e^{-r\tau(\theta_{(1)},\theta_{(2)})} S(s(\theta_{(2)}), P_{\tau(\theta_{(1)},\theta_{(2)})})] + cNF(\hat{\theta}) \qquad (41)\\
&= \int_{\underline{\theta}}^{\hat{\theta}} \int_{\underline{\theta}}^{\theta_2} N(N-1)\mathbb{E}[e^{-r\tau(\theta_1,\theta_2)} S(s(\theta_2), P_{\tau(\theta_1,\theta_2)})|\theta_1, \theta_2] f(\theta_1) f(\theta_2)[1-F(\theta_2)]^{N-2} d\theta_1 d\theta_2 + cNF(\hat{\theta}).
\end{aligned}$$

Differentiating this w.r.t. c, we obtain

$$\left[N(N-1)\int_{\underline{\theta}}^{\hat{\theta}} \mathbb{E}[e^{-r\tau(\theta_1,\hat{\theta})} S(s(\hat{\theta}), P_{\tau(\theta_1,\hat{\theta})})] f(\theta_1) d\theta_1 f(\hat{\theta})[1-F(\hat{\theta})]^{N-2} + cNf(\hat{\theta})\right] \frac{d\hat{\theta}}{dc} + NF(\hat{\theta}).$$

At $c = 0$, the first term is zero because $S(s(\hat{\theta}), P_{\tau(\theta_1,\hat{\theta})}) = 0$ as we have just shown. The derivative is positive, so a small entry fee leads to an increase in revenue. The argument for FPAs is almost identical, except for an additional term reflecting how the present value of the security a type changes when the boundary of the differential equation for the bidding strategy changes, which contributes positively to the derivative with the assumption. Note since $\frac{P_0^\beta(\beta-1)^{\beta-1}}{\beta^\beta\hat{\theta}^{\beta-1}} - X \geq 0$,

$$\frac{d\hat{\theta}}{dc} = -[1-F(\hat{\theta})]^{2-N}\left[(1-F(\hat{\theta}))\frac{P_0^\beta(\beta-1)^\beta}{\beta^\beta\hat{\theta}^\beta} + (N-1)f(\hat{\theta})\left(\frac{P_0^\beta(\beta-1)^{\beta-1}}{\beta^\beta\hat{\theta}^{\beta-1}} - X\right)\right]^{-1} < 0.$$

Now the social welfare in SPA is given by

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{\theta_{(1)}<\hat{\theta}\}}(e^{-r\tau(\theta_{(1)},\theta_{(2)})}(P_{\tau(\theta_{(1)},\theta_{(2)})} - \theta_{(1)}) - X)]\\
&= \int_{\underline{\theta}}^{\hat{\theta}} \int_{\underline{\theta}}^{\theta_2} N(N-1)\mathbb{E}[e^{-r\tau(\theta_1,\theta_2)}(P_{\tau(\theta_1,\theta_2)} - \theta_1) - X|\theta_1, \theta_2] f(\theta_1) f(\theta_2)[1-F(\theta_2)]^{N-2} d\theta_1 d\theta_2\\
&\quad + [1-F(\hat{\theta})]^{N-1} N \int_{\underline{\theta}}^{\hat{\theta}} \mathbb{E}[e^{-r\tau^*(\theta_1)}(P_{\tau^*(\theta_1)} - \theta_1) - X|\theta_1] f(\theta_1) d\theta_1 \qquad (42)
\end{aligned}$$

Differentiating this w.r.t. c, we obtain

$$\frac{d\hat{\theta}}{dc}\Big([1-F(\hat{\theta})]^{N-1}N\mathbb{E}[e^{-r\tau^*(\hat{\theta})}(P_{\tau^*(\hat{\theta})}-\hat{\theta})-X|\hat{\theta}]f(\hat{\theta})$$
$$+\int_{\underline{\theta}}^{\hat{\theta}}N(N-1)\mathbb{E}[e^{-r\tau(\theta_1,\hat{\theta})}(P_{\tau(\theta_1,\hat{\theta})}-\theta_1)-e^{-r\tau^*(\theta_1)}(P_{\tau^*(\theta_1)}-\theta_1)|\theta_1]f(\theta_1)d\theta_1 f(\hat{\theta})[1-F(\hat{\theta})]^{N-2}\Big)$$

The first term in the bracket is $Ncf(\hat{\theta})$ by the definition of the cutoff type. It is the expected additional fee collected when the cutoff type increases marginally. The second term is negative and $\mathbb{E}[e^{-r\tau(\theta_1,\hat{\theta})}(P_{\tau(\theta_1,\hat{\theta})}-\theta_1)-e^{-r\tau^*(\theta_1)}(P_{\tau^*(\theta_1)}-\theta_1)|\theta_1]$ gives a measure of the welfare loss when the winning bidder has to pay a security bid by the cutoff type. In general the sign is indeterminate. However, at $c=0$, there would not be any moral hazard if the second best type is the cut-off type, and the derivative is zero. The second derivative at $c=0$ has indeterminate sign. It can be shown that for equity auctions with uniform distributions and for $\beta>2$, there is always an N such that the second derivative is positive, indicating that the welfare maximizing auction involves an entry fee. In general, the effect of entry fee on social welfare is mixed and depends on the assumptions of the distribution and parameter values. The situation is similar in FPAs. □

P. Proof of Proposition 7.1

Proof. First consider SPAs. The revenue is

$$\begin{aligned}\mathbb{E}[e^{-r\tilde{\tau}}S(s(\theta_{(2)}),P_{\tilde{\tau}})\mathbf{1}_{\{\theta_{(2)}\leq\hat{\theta}\}}]&=\mathbb{E}[\big(e^{-r\tilde{\tau}}(P_{\tilde{\tau}}-\theta_{(1)})-U(\tilde{\tau},s(\theta_{(2)}),\theta_{(1)})\big)\mathbf{1}_{\{\theta_{(2)}\leq\hat{\theta}\}}]\\ \leq&\mathbb{E}[\big(e^{-r\tilde{\tau}}(P_{\tilde{\tau}}-\theta_{(1)})-X\big)\mathbf{1}_{\{\theta_{(1)}\leq\hat{\theta}\}}]\equiv R_0\end{aligned} \tag{43}$$

where $\tilde{\tau}=argmax_\tau U(\tau,s(\theta_{(2)}),\theta_{(1)})$ and $U(\tau,s,\theta)=\mathbb{E}[e^{-r\tau}(P_\tau-S(s,P_\tau)-\theta)]$. Similarly in FPAs, the revenue is bounded above by R_0 with $\tilde{\tau}=argmax_\tau U(\tau,s(\theta_{(1)}),\theta_{(1)})$. Let s_w denote the index the winning bidder pays in general. Then in FPAs and SPAs, the revenue is bounded above by R_0 with $\tilde{\tau}=argmax_\tau U(\tau,s_w,\theta_{(1)})$

The revenue from cash auction would be the expected second highest valuation $R_2\equiv\mathbb{E}[(W(\theta_{(2)})-X)\mathbf{1}_{\{\theta_{(2)}\leq\hat{\theta}\}}]$. When $N\to\infty$, $\theta_{(2)}-\theta_{(1)}\overset{a.s.}{\to}0$. Thus $W(\theta_{(2)})-W(\theta_{(1)})\overset{a.s.}{\to}0$. Now $\mathbf{1}_{\{\theta_{(2)}\leq\hat{\theta}\}}$ and the above are bounded, by bounded convergence, R_2 converges a.s. to $R_1\equiv\mathbb{E}[(W(\theta_{(1)})-X)\mathbf{1}_{\{\theta_{(1)}\leq\hat{\theta}\}}]$.

If $R_1 - R_0$ converges to a quantity bounded below by a positive constant, the claims follow. First note $U(\tau, s_w, \theta_{(1)})$ admits an optimal stopping solution involving threshold strategies. To see this, write $U(\tau, s_w, \theta_{(1)}) = D(P_0; P)[P - \theta_{(1)} - \sum_{i \in I} a_i(s_w)[P - b_i(s_w)]^+]$, which admits a maximizer $\tilde{P}(\theta_{(1)})$. Then use that as an investment trigger and apply the standard verification argument. Next, as $\theta_{(1)} - \underline{\theta} \stackrel{a.s.}{\rightarrow} 0$, the investment trigger in cash auctions converges to $P^* = \frac{\beta}{\beta-1}\underline{\theta}$, and $\tilde{P}(\theta_{(1)})$ to $\tilde{P}^* = \tilde{P}(\underline{\theta})$. Whether $\tilde{P}^* \in [b_m, b_{m+1})$ or not, $|\tilde{P}^* - P^*| \geq M$. Since P^* is the optimal trigger for $\mathbb{E}[e^{-r\tau}(P_\tau - \underline{\theta})]$, $R_1 - R_0 \stackrel{a.s.}{\rightarrow} \epsilon$ for some $\epsilon > f(M)$, where $f(M)$ is a function of M that is positive and independent of N. This basically says moral hazard is persistent.

Therefore as N becomes big, R_2 converges to R_1 dominates R_0 in the limit. Thus cash auctions yield higher revenue than the security-bid auctions.

Finally, since cash auctions are efficient and regular security involves moral hazard cost that is persistent when N becomes large, cash also dominates in terms of welfare. □

Figures

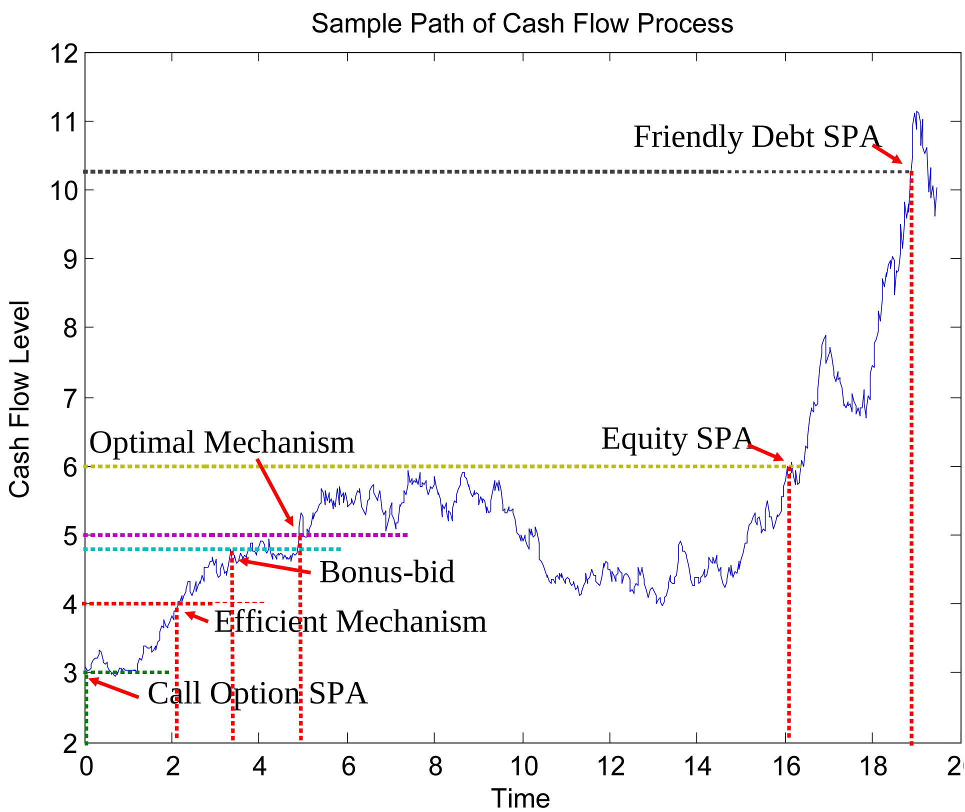
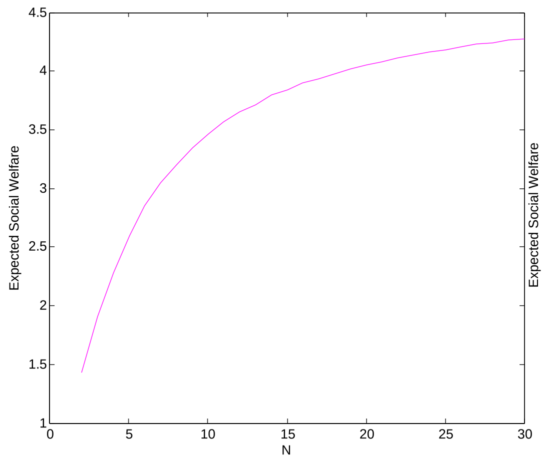
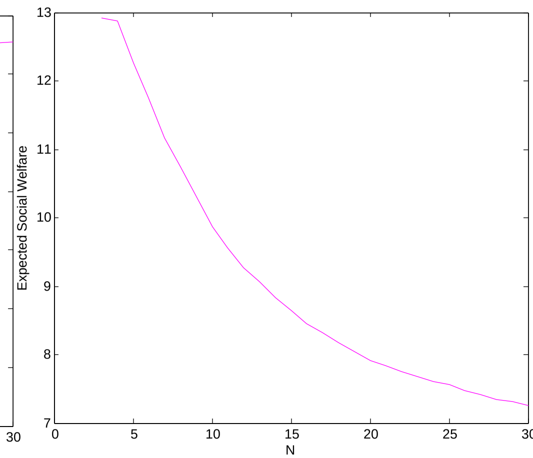


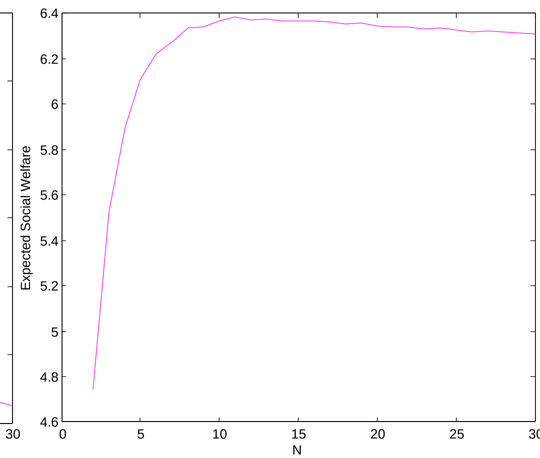
Figure 1: Investment Thresholds under Various Mechanisms. Simulated with $\mu = 0.06$, $\sigma = 0.2$, $r = 0.16$, $\theta \sim Unif[1.5, 5]$, $X = 0.4$, $P_0 = 3$. Horizontal dotted lines indicate investment triggers, vertical lines indicate investment times.



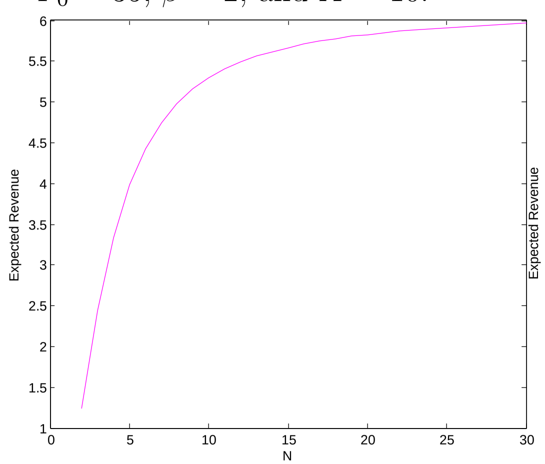
(a) Welfare with $\theta \stackrel{d}{=}$ Unif$[20, 50]$, $P_0 = 35$, $\beta = 2$, and $X = 10$.



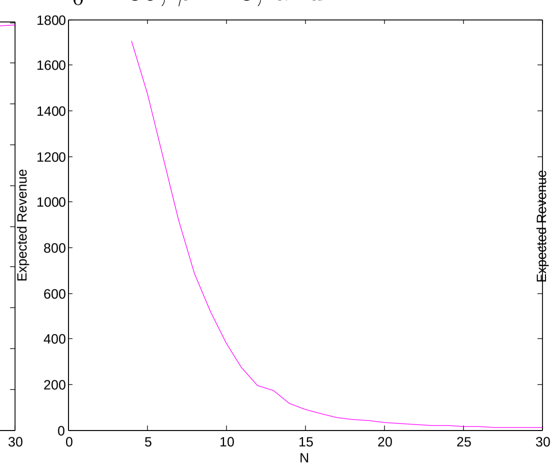
(b) Welfare with $\theta \stackrel{d}{=}$ Unif$[20, 50]$, $P_0 = 35$, $\beta = 8$, and $X = 1$.



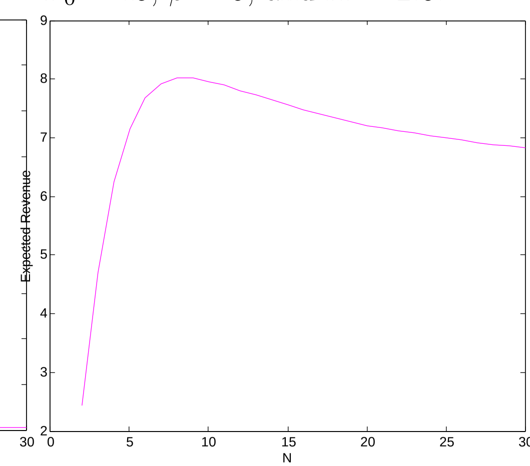
(c) Welfare $\theta \stackrel{d}{=}$ Unif$[30, 60]$, $P_0 = 45$, $\beta = 5$, and $X = 2.5$.



(d) Rev with $\theta \stackrel{d}{=}$ Unif$[30, 60]$, $P_0 = 45$, $\beta = 5$, and $X = 2.5$

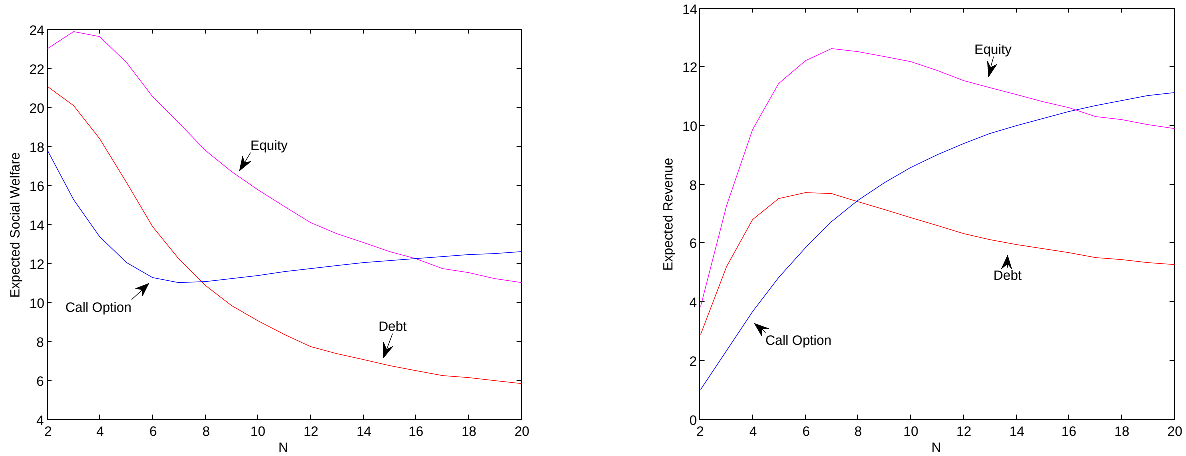


(e) Rev with $\theta \stackrel{d}{=}$ Unif$[20, 50]$, $P_0 = 35$, $\beta = 25$, and $X = 0.1$



(f) Rev with $\theta \stackrel{d}{=}$ Unif$[20, 50]$, $P_0 = 35$, $\beta = 8$, and $X = 1$.

Figure 2: Impact of N on Social Welfare and Seller's Revenue
One million simulations in SPA with equity bids.



Expected Social Welfare — Expected Revenue

Figure 3: Impact of N on Social Welfare and Seller's Revenue for SPAs with Equities, Friendly Debts and Call Options.
One million simulations with $\theta \overset{d}{=}$ Unif. $[20, 50]$, $P_0 = 35$, $\beta = 10$, and $X = 1$.

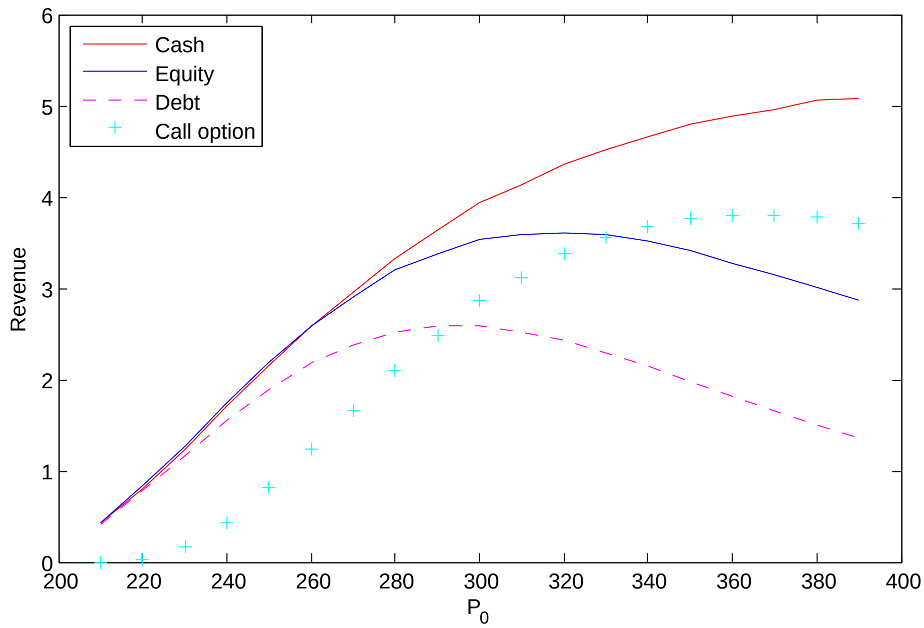


Figure 4: Revenues (normalized) for SPAs, one million simulations for $\theta \sim Unif[200, 500]$, $\beta = 5$, $X = 10$, $N = 5$, $P_{00} = 210$.

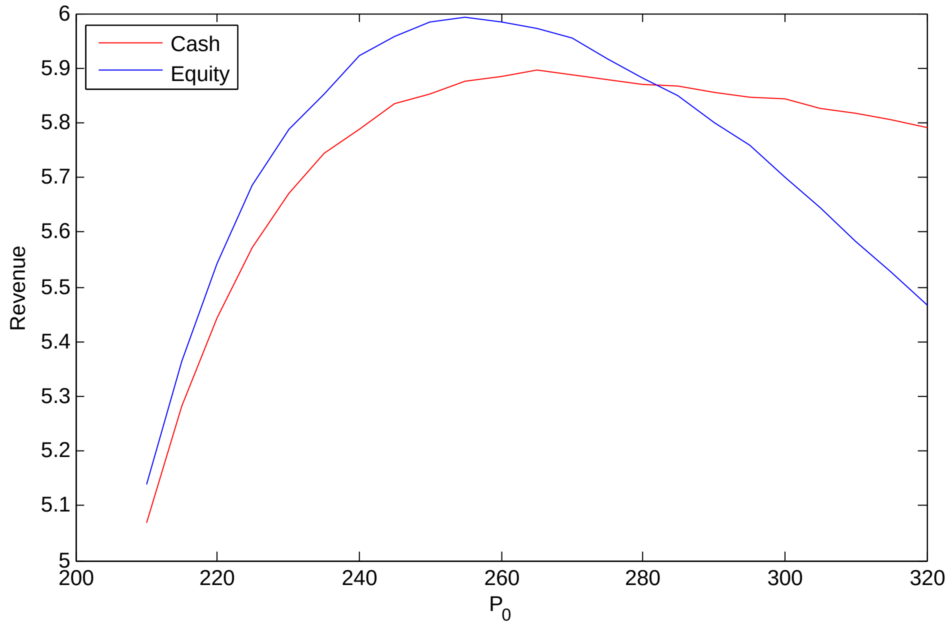


Figure 5: Revenues (normalized) for FPAs, one million simulations for $\theta \sim Unif[200, 500]$, $\beta = 5$, $X = 0$, $N = 5$, $P_{00} = 210$, and entry fee $c = 8$.

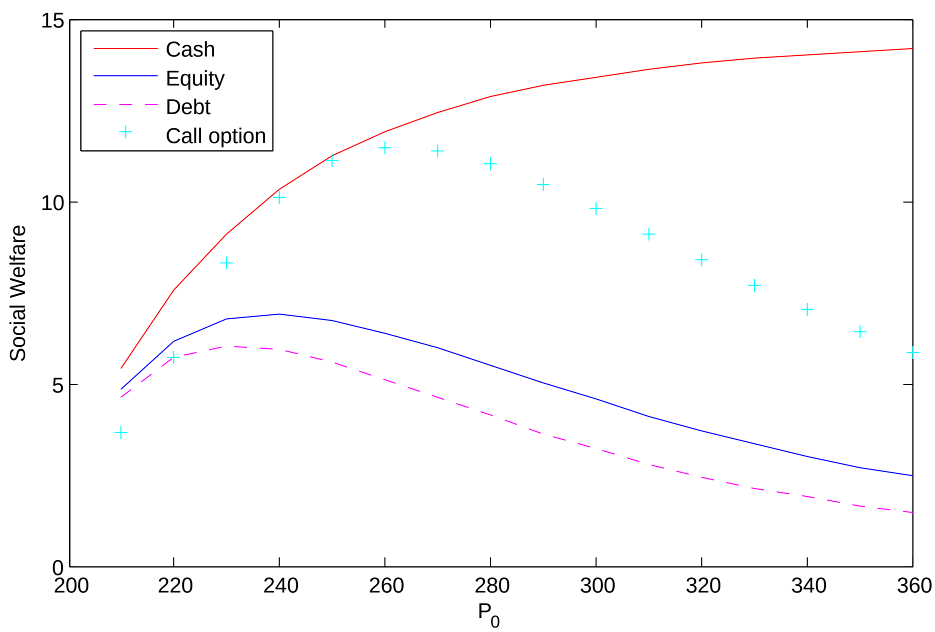


Figure 6: Social Welfare (normalized) for SPAs, one million simulations for $\theta \sim Unif[200, 500]$, $\beta = 6$, $X = 8$, $N = 30$, $P_{00} = 210$ and entry fee $c = 2$.