Large Orders in Small Markets: On Optimal Execution with Endogenous Liquidity Supply

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Abstract

Increased intermediation made some investors “too large” for their markets. If such investor needs to sell quickly, then he cannot reach buyers who arrive later. Market makers then supply liquidity by taking on inventory to sell to future buyers. We solve a Stackelberg game where a large uninformed seller executes optimally, fully cognizant of the response of Cournot-competitive market makers. The game therefore endogenizes both demand and supply of liquidity. The closed-form solution yields several insights. First, stealth trading is both privately and socially costly because market makers incur additional cost not knowing when execution ends. Second, the presence of a large seller does not unambiguously benefit other participants. Market makers benefit only if there is enough risk-absorption capacity or if the execution period is short. Other investors benefit only when the seller sells at high enough intensity. Under sunshine trading where market makers know when execution ends price pressure might subside before execution ends rationalizing such pattern observed in the data. Our normative results have direct implications on institutional investors: it allows them to determine how much they are able to trade in a “small market” within a particular time frame, and to assess the dependence of the implied optimal trade intensity on market conditions, such as the rate of investor arrivals, the demand elasticity of these investors, the number of market makers, and funding costs.

The past few decades have seen a gradual rise in intermediated investment in securities markets. French (2008, Table 1) for example reports that direct holdings of U.S. equity declined from 47.9% in 1980 to 21.5% in 2007.[1] This has given rise to large institutional investors for whom asset reallocations trigger liquidity demands that are large relative to the markets they trade in. This paper studies liquidity in this context. We study the optimal execution problem of a strategic large investor who understands that the market makers who supply liquidity to him optimally respond to his strategy. Let us first frame this setting appropriately in the context of the extant literature on liquidity.[2]

In a nutshell, liquidity is the market for immediacy. Demanders pay a premium for buying or selling *now*. This premium compensates suppliers who enter a costly position to offload it to future end-user investors. Grossman and Miller (1988), henceforth referred to as GM88, is a classic model in this literature that offers an economic analysis of the size of this premium. Their model features two periods and a finite number of (price-taking) market makers.

We generalize the GM88 setting with particular emphasis on *time*, replacing the two periods by continuous time. More specifically, the liquidity demander enters at time zero and is (continuously) present for a finite period of time (in this sense he demands immediacy). In other words, he needs to be done trading by a time that is private information to him. He enters with an extremely large sell order, in a sense larger than what the (small) market can supply liquidity for. He therefore rations the quantity he sells in an optimal way to maximize proceeds. Such optimal execution by the large seller replaces the exogenous sell order in the first period in GM88.

In the model, the natural counterparty to the large seller's trades are end-user buyers who arrive according to a Poisson process. These random arrivals replace the GM88 buyers who (only) arrive in the second period. Similar to GM88, there is a finite number of market makers connecting the large seller with the buyers. Our model additionally features stochastic arrivals of end-user sellers. Importantly, the quantity that sellers and buyers demand upon arrival depends on the bid and ask prices posted by market makers.

Contrary to GM88, the market makers in our model are Cournot competitors (not price takers). They earn an endogenous liquidity premium by, on average, net buying in the period when the large seller executes and net selling to end users arriving later. They effectively smear out the large seller's short-term sells to long-term natural buyers.

The *key* contribution relative to GM88 is that liquidity demand is endogenized, in addition to liquidity supply. To do so in a tractable way, we assume that the large seller is a Stackelberg leader in the game with market makers. He fixes (and commits to) a constant optimal sell intensity

[1]The Economist reports an even more dramatic trend with individual-investor ownership of U.S. equity declining from 90% in 1950 to 30% in 2014. "Reinventing the Deal," The Economist, Oct 24, 2015.

[2]Examples of articles surveying the literature are: Madhavan (2000), Vayanos and Wang (2013), and Amihud, Mendelson, and Pedersen (2005). The most recent literature has focused on high-frequency traders (HFTs) as the "new" market makers. Example surveys of this recent literature are: Jones (2013), O'Hara (2015), and Menkveld (2016).

until his departure.[3] The innovation is that we solve for the optimal sell intensity where the seller internalizes the optimal response of market makers.

More generally, we contribute to a large literature where thus far either liquidity demand or liquidity supply was endogenized, but typically not both.[4] The following papers are the exceptions. Vayanos (2001) studies a single uninformed investor who trades strategically. His order size is unknown to market makers who are price takers. Pritsker (2009) studies strategic trading of agents who are aware of future trade needs of a distressed seller. Goettler, Parlour, and Rajan (2005) present a numerical analysis of a limit-order market where agents endogenously decide to post a limit order (i.e., supply liquidity) or take a limit order (i.e., demand liquidity). Gabaix et al. (2006) sketches the contours of a model that explains why investor size distribution is more fat tailed than the optimal order sizes they pick and therefore the price impact distribution. They explicitly invite others to develop a complete dynamic model (p. 487): "We leave to future research the important task of modeling the specifics of the cascade that followed the initial impulse." (An invitation we gladly accepted.) Choi, Larsen, and Seppi (2018a) add a strategic uninformed portfolio rebalancer to a Kyle (1985) type setting with a strategic informed investor, noise traders, and competitive risk-neutral market makers. Finally, Choi, Larsen, and Seppi (2018b) study the equilibrium interaction between multiple strategic investors with different trading targets but the same trading horizon. In contrast, the strategic agents in our model have different horizons as we focus on the time dimension of liquidity.

Our contribution relative to these papers is a fully tractable model that features a large seller who understands the equilibrium response of strategic market makers. The model yields an endogenous price impact function that includes a bid-ask spread, depth at both the bid and the ask quote, and price pressure (i.e, the wedge between the midquote and fundamental value) due to market maker inventory. It is the latter that large investors arguably care much about when they submit repeated buy or sell trades during the execution of a large order.

In addition to contributing academically, we believe that the paper will be of interest to industry professionals, regulators, and retail investors. Institutional investors are likely to benefit from the paper's normative results. How much is one able to trade in a "small market" within a particular time frame? How does the implied optimal trade intensity depend on market conditions (i.e., the model's deep parameters, e.g., the rate of investor arrivals, the demand elasticity of these investors, the number of market makers, or funding costs)? It is for these same reasons that the study

[3]This simplification buys us tractability. It is not an innocuous assumption as it excludes, for example, the seller being able to reduce his sell intensity after observing fewer-than-expected buyer arrivals. Execution at fixed intensity however seems to occur often in real-world markets where VWAP execution strategies are popular (for a detailed argument see the end of Section 1.)

[4]Examples of the former are the optimal execution models of Bertsimas and Lo (1998), Almgren and Chriss (2001), Huberman and Stanzl (2005), Obizhaeva and Wang (2013), Gatheral and Scheid (2011), Boulatov, Bernhardt, and Larionov (2016), and van Kervel, Kwan, and Westerholm (2018). Examples of the latter other than GM88 are Amihud and Mendelson (1980), Ho and Stoll, Hans R. (1981), Weill (2007), Hendershott and Menkveld (2014), and Bank, Ekren, and Muhle-Karbe (2018).

should also interest those who offer optimal execution services such as Goldman Sachs (GSET), BlackRock (Aladdin), or ITG (ACE).[5]

The paper's analytic results on how large orders affect all market participants is of interest to regulators and retail investors. The effects of large orders on Poisson investors speak to retail investors' interests, and to regulators who often have a mandate to protect them. More generally, the fine description of how all agents are affected by the presence of a large order enables them to assess market quality, which necessarily concerns all market participants (i.e., welfare). And, specifically, the paper's results on "price instabilities" caused by large orders are high on the agenda of regulators who worry about sizable liquidity costs. A prominent example is the recent SEC regulation that makes open-end funds assess the liquidity risk of their holdings. Funds need to report "days-to-cash" thus emphasizing the time dimension of liquidity that is at the heart of our model (SEC 2016).[6]

Another example is the cover-2 capital requirement regulators set for central counterparties (CCPs). This requirement forces CCPs to assess the liquidity premium paid for unwinding the positions that they inherit from two failed accounts. They need to do so in a pre-specified *close-out* period.

Results. The model yields several results. First, there is a *unique* symmetric equilibrium in pure strategies where intermediaries act as market makers. Note that this result differs markedly from Brunnermeier and Pedersen (2005) where predatory-trading is the unique equilibrium. We believe that the *key* difference between their setting and ours that explains this result is that large sellers' trades have permanent price impact in their setting, not in ours. This combined with their *assumed* exogenous "transitory price impact" (p. 1831) rules out prices immediately dropping to reflect the long-term impact in equilibrium. The price decline therefore has to be smooth which explains why in their setting, predatory trading becomes profitable and, in fact, the unique equilibrium strategy.

In our setting, in contrast, the *transitory price impact* is endogenous and derived from first principles. Large orders command a short-term price impact to compensate market makers, but no long-term price impact as eventually enough buyers will be found (although this could take a very long time). In such setting, selling along with the large seller initially to prey on his trades is inherently costly as, in equilibrium, one sells below fundamental value (i.e., the long-term price). In sum, the Brunnermeier and Pedersen (2005) setting is one of "stressed" sells where the position needs to land on finitely many investors in finite time so that the price has to adjust permanently.

[5]As a matter of fact, ACE is based on an (undisclosed) structural model. Investment Technology Group (ITG) (2007) explicitly advertises its advantage by stating: "The ITG ACE model is not a purely econometric model calibrated based on transaction cost data. Rather, it is a structural model that uses parameters estimated econometrically." Grinold and Kahn (1999, Ch. 15) emphasize the importance of a structural dynamic model for transactions cost in their comprehensive analysis of active portfolio management.

[6]For each order duration, the model yields an implied optimal sell intensity. The product of duration and intensity yields the optimal amount sold. Working backwards, if a seller is given an amount to sell, then these results imply a duration for which this amount was optimal and therefore a days-to-cash result.

In our setting, investors are in infinite supply albeit spaced out across an (infinitely) long period.[7]

Second, stealth trading by the large seller is costly. More precisely, if market makers observe or, rather, experience his sell intensity but do not know when it ends then the large seller pays a larger liquidity premium. The counterfactual here is sunshine trading where the large seller reveals when his execution ends. The larger premium compensates market makers for the uncertainty about how long the directional flow will last. Our calibration to actively traded stocks shows that this premium can be sizeable and amount to 4.2% for large-cap stocks.[8]

Third, not only is there a direct cost associated with stealth trading (previous point), there is also an indirect cost because hiding requires the large seller to use the same trade intensity for all of his durations (as otherwise market makers learn). Unlike stealth, sunshine trading allows the large seller to make the intensity depend on duration (we discuss how to think about sunshine trading in practice at the end of the introduction). And, indeed, solving the model we find that the optimal intensity differs across durations. More precisely, it strictly decreases in duration. Short trades execute at high intensity, long trades at low intensity. The calibration shows that this indirect effect raises net proceeds by another 1.3% so that the direct and indirect effect combined amount to 4.2+1.3=5.5%.

Fourth, zooming out of the large seller, do others benefit from the large seller's presence? Do market makers benefit? Do end-users benefit? The model yields some non-trivial results. The presence of a large seller benefits market makers as competition is soft and guaranteed flow is a source of rents for them. However, this result *only* holds if the risk-absorption capacity (henceforth referred to as risk-capacity) of the market is large enough or if the large-seller's order is of short enough duration. The risk-capacity condition is met when, for example, market makers are risk tolerant, there are many of them, or if investors arrive frequently. The risk-capacity condition holds in the calibrated model and the market makers value function is 7.6% higher when there is a large seller relative to the baseline of no large seller (for the case of sunshine trading).

For other investors the result again is a conditional one. Only if the large seller trades at sufficiently high intensity do they benefit. The model exposes the channels that lead to this finding. The presence of a large seller increases the bid-ask spread in equilibrium and causes negative price pressure. The spread effect hurts investors but the pressure benefits them as they, on average, buy at depressed prices. If the large seller trades at a high enough intensity, then the pressure effect dominates. The intuition is that market makers' position increases rapidly in such case and since price pressure scales with it, investors enjoy higher subsidies on their net buying. In the calibration, the large seller trades intensively enough as investors' realized surplus relative to the baseline is 11.5% higher.

[7]There exist other studies that, after endogenizing the transitory price impact, find no predatory trading in equilibrium. Choi, Larsen, and Seppi (2018b) and Pritsker (2009) discuss predatory trading in their setting and either conjecture it not to be there (former) or find its effect to be small (latter).

[8]The large seller is assumed to truthfully reveal his duration in the sunshine trading case. We stopped short of micro founding such assumption, but believe it could be done by extending the model to a repeated game with appropriate punishment. This however is out of scope here.

Fifth, we solve a social planner's problem to gauge how efficient the sunshine equilibrium is. The solution yields a constrained first-best as we allow the planner to only set prices in the second stage of the Stackelberg game. We believe that this is the most natural planner problem to solve since prices can in principle be regulated (e.g., through circuit breakers), but agents' trade decisions cannot. In this planner problem, prices turn out to be more responsive to market making inventories. The reason is that the planner internalizes the benefit that such pressure creates for investors. As a result, inventories mean-revert more quickly and this reduces the deadweight loss imposed by market makers carrying inventory. The calibration shows that the social inefficiency (i.e., the wedge between welfare in sunshine and the planner problem) is relatively small: 1.4%. An important reason for this is that there are many market makers. A monopoly or duopoly would cost the market dearly: 24.4% and 11.1%, respectively.

Sunshine trading in practice. Do uninformed investors engage in sunshine trading in practice? Admati and Pfleiderer (1988) are the first to analyze sunshine trading and find that the portfolio insurance company of Leland, O'Brien, and Rubinstein was the first to engage in it. Admati and Pfleiderer, however, consider sunshine trading as pre-announcing a large trade a few hours before it takes place. When discussing real-world markets, they point out that although sunshine trading may theoretically be optimal for uninformed traders, regulation strains such practice as it is hard to distinguish it from "prearranged trading" which is illegal.

One could argue that this issue has become mute in today's electronic markets. A large seller does not need to pre-announce because he can effectively communicate duration by making trade intensity depend on duration. By observing intensity market participants are able to learn duration, perfectly so if the relationship is strictly monotonic. This happens to be the case for optimal execution in the case of sunshine trading where intensity is low for long durations and high for short durations. This negative monotonic relationship appears to supported by evidence on realized executions (e.g., Zarinelli et al. 2015, Fig. 2 or Brough 2010, Tab. 2.6).

Perhaps the most discriminatory empirical support for our model is the finding in Zarinelli et al. (2015, Fig. 8) that price pressure subsides before an institutional trader stops executing. This pattern arises endogenously in the sunshine equilibrium because market makers know when execution ends and find it optimal to reduce price pressure before this ending time. This pattern is absent in the stealth trading equilibrium. More importantly, this sets our model apart from other optimal execution models where liquidity supply is exogenous (hence the term "discriminatory evidence").

1 Model primitives

The model primitives capture a setting where a large seller maximizes net proceeds and accounts for market makers' equilibrium response to his execution strategy.[9] The seller is large in the sense that he has an unlimited supply of the same security to sell and effectively only faces a time constraint. Market makers are Cournot competitors who absorb the seller's flow to resell to stochastically arriving end-user buyers and sellers. In a nutshell, the seller exerts continuous pressure on the market makers who only occasionally are able to mean-revert inventory by the lumpy arrival of end-users. We believe this is a reasonable description of a large orders in small markets.

Before describing the primitives in detail, we believe it is useful to illustrate the environment graphically. Figure 1 provides such illustration. The top graph depicts trades where the large seller sends a continuous flow to the market up until d which is his time constraint (red line). End-user investors arrive at random points in time and trade an amount (black line) that depends on where bid and ask prices are relative to fundamental value. A buyer, for example, buys more when the bid price is further below fundamental.

The middle graph illustrates how the position of market makers evolves with time. As markets need to clear, their positions equal the negative cumulative net flow from the top graph. Notice how the market maker position increases linearly due to the continuous net selling by the large seller until it jumps down when an end-user buyer happens to arrive. Then the market maker position continues to grow again until it jumps up due to an end-user seller arriving and offloading inventory.

The bottom graph illustrates the bid and ask price relative to fundamental. The bid and ask are both skewed downwards from the very start as market makers observe the sell intensity and respond optimally. The middle of the bid and ask quote, henceforth referred to as the midquote, becomes pressured downwards. The negative pressure means that if an end-user buyer arrives, then he demands a large quantity as the low ask makes it relatively cheap to buy. An end-user seller on the other hand sells less on arrivals as the low bid quote makes it unattractive to sell. Note that indeed in the top graph the trade size of buy orders is larger than what it is for sell orders.

The remainder of this section describes the model primitives in full detail. It requires a substantial amount of notation. To keep track of it all, Appendix A provides a summary of all notation used in the paper. The model is set in continuous time, features a single security and three types of agents: a large seller, N market makers, and stochastically arriving buyers and sellers.

Security. The security's fundamental (common) value follows a Brownian motion with volatility $\sigma > 0$:

$$\mathrm{d}m_t = \sigma \mathrm{d}B_t. \tag{1}$$

[9] The symmetric case of a large buyer can be treated using completely symmetric arguments.

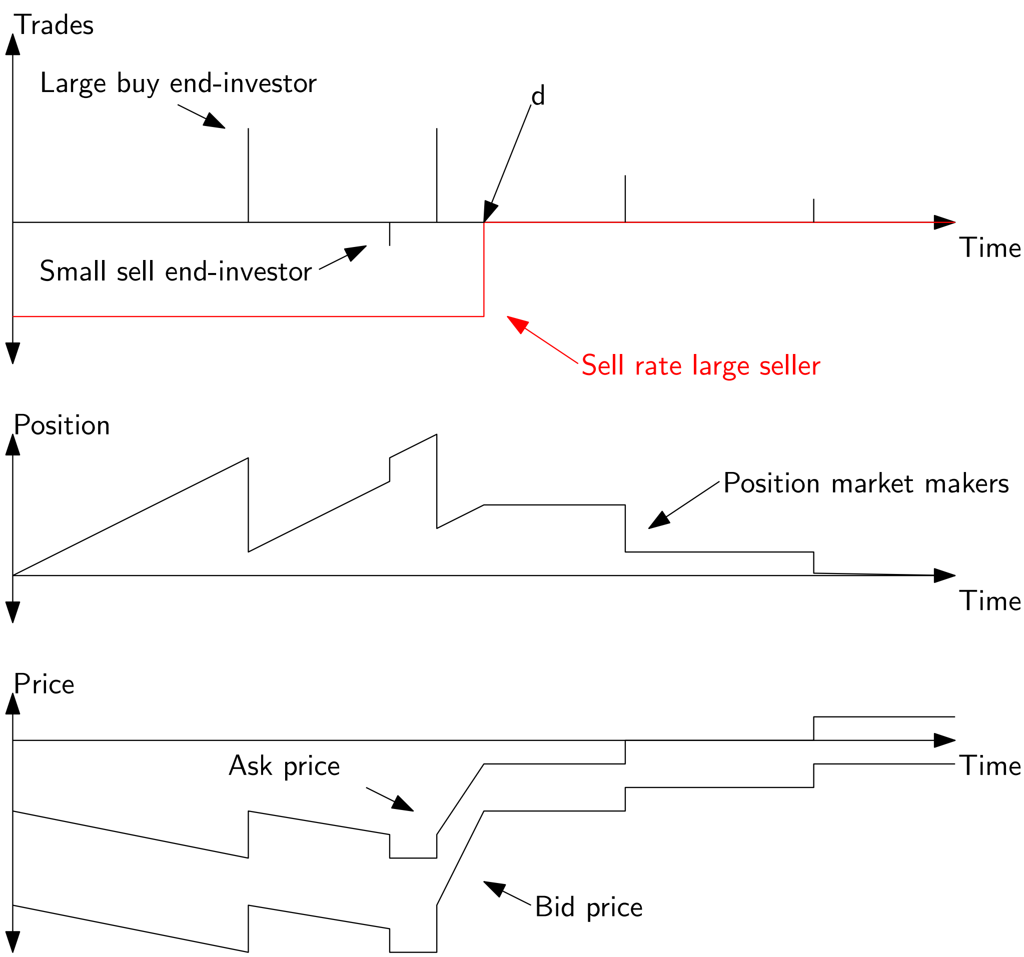

Figure 1. Model visualization. This figure illustrates the model by plotting a realization of the various quantities. The top graph plots the (continuous) net flow of the seller as well as discrete arrivals of buyers and sellers. The middle graph plots market makers' positions which result from the net flows in the top graph. The bottom graph plots the bid and ask prices relative to fundamental value.

The fundamental value is common knowledge to all agents in the model and therefore not a source of asymmetric information.

End-user investors. End-user investors are modeled in reduced form. They consist of either buyers or sellers. Buyers arrive according to a Poisson process with rate λ. Let N_t^B denote the cumulative number of buyers arriving in the interval $[0, t]$. Upon arrival the quantity they wish to buy depends on how much the ask price is below their (relative) reservation value ω:

$$q_t^a := q^a(p_t^a) = \delta\left((m_t + \omega) - (m_t + p_t^a)\right) = \delta\left(\omega - p_t^a\right), \tag{2}$$

where the scalar δ is the price sensitivity of demand and p_t^a is the ask quote minus the fundamental value m_t at time t. Without loss of generality, p_t^a is defined as the relative ask quote and, for short, will be referred to as ask quote throughout. This avoids unnecessary clutter in mathematical expressions. Similarly, p_t^b is defined as m_t minus the bid quote (and therefore is expected to be mostly negative).

Sellers are modeled symmetrically. N_t^S denotes the cumulative number of sellers by time t which also follow a Poisson process with rate λ. The quantity they wish to sell at the bid is:

$$q_t^b := q^b(p_t^b) = \delta\left(p_t^b + \omega\right), \tag{3}$$

where $-\omega$ is the relative reservation value of sellers. The Poisson processes are assumed to be mutually independent, and independent of the Brownian motion driving the fundamental value process.

Large seller. The large seller arrives at time zero and submits an order of duration d. He strategically chooses and commits to a deterministic sell intensity f that maximizes his net proceeds. These proceeds are defined as the trading revenue minus his reservation value, where the relative reservation value for the large seller is assumed to be $-\omega$, thus equal to that of end-user sellers:

$$P_d(f) = \int_0^d e^{-\beta t} f\left(\omega + p_t^b\right) \mathrm{d}t. \tag{4}$$

The large seller sets the intensity such that net proceeds are maximized:

$$\arg\max_f \mathbb{E}\left[P_d(f)\right] \tag{5}$$

Note that f is unconstrained by assumption as the seller has an infinite supply of the securities. This is by construction and focuses the analysis on a seller who is truly large relative to the securities markets he trades in.

Importantly, d is private information to him but it is common knowledge that d is drawn from an exponential distribution with parameter ν (i.e., its mean is $1/\nu$). d therefore is the only source of asymmetric information in the model. Revelation of this quantity turns out to be a first-order driver of the equilibrium. We will return to this issue when discussing two types of equilibria in Section 2.1: "sunshine" and "stealth." The equivalents of (5) in these two types are (36) and (37), respectively.

Market makers. The market makers are Cournot competitors who trade in continuous time. They decide how much to offer at each candidate bid or ask quote. Denote by $q^a_{jt} := q^a_{jt}(p^a_t)$ the amount market maker j offers at the ask price p^a_t at time t. Then the total quantity offered at that price is $\sum_{n=1}^N q^a_{nt}$. Market clearing requires the amount an end-user buyer desires upon arrival to equal the amount offered to him at the ask price:

$$q^a_t = \sum_{n=1}^{N} q^a_{nt}. \tag{6}$$

This market-clearing condition combined with (2) therefore nails the ask price as a function of the quantities that market makers offer. A similar derivation holds for the bid price and the resulting expressions for the equilibrium ask and bid price are, respectively:[10]

$$p^a_t = \omega - \frac{1}{\delta}\sum_{n=1}^{N} q^a_{nt} \quad \text{and} \quad p^b_t = \frac{1}{\delta}\sum_{n=1}^{N} q^b_{nt} - \omega. \tag{7}$$

These pricing equations relate equilibrium prices to the quantities offered by market makers that is likely to depend on their inventory state which evolves as follows:

$$\mathrm{d}i_{jt} = 1_{\{t\le d\}}\frac{f}{N}\mathrm{d}t - q^a_{jt}\mathrm{d}N^B_t + q^b_{jt}\mathrm{d}N^S_t, \tag{8}$$

where 1_A is the indicator function that is one when A holds and zero otherwise. The first term captures the symmetric sharing of the large seller's flow in the period that the large seller is executing.

A market maker's wealth changes based on his ability to sell at (ask) prices above fundamental value or buy at (bid) prices below fundamental values. Wealth is defined as trading revenues relative to the fundamental. Formally, a market maker's (gross) wealth therefore evolves as:

$$\mathrm{d}W_{jt} = \left(\omega - \frac{1}{\delta}\sum_{n=1}^{N} q^a_{nt}\right) q^a_{jt}\mathrm{d}N^B_t - \left(-\omega + \frac{1}{\delta}\sum_{n=1}^{N} q^b_{nt}\right)\left(q^b_{jt}\mathrm{d}N^S_t + 1_{\{t\le d\}}\frac{f}{N}\mathrm{d}t\right). \tag{9}$$

The disutility of a non-zero position appears explicitly in the market maker's objective function as

[10] Note that the total depth at the ask price is q^a_t. At the bid price the total amount is q^b_t. Note that the bid also contains the amount sold to the large seller but that is a vanishingly small amount (i.e., $f\mathrm{d}t$).

a pecuniary cost that scales with inventory squared. A market maker's value function can now be defined explicitly as:

$$v_j^M(i,f,t) = \sup_{q^j|q^{-j}} \mathbb{E}\left[\int_t^\infty e^{-\beta(s-t)}\left(\mathrm{d}W_{js} - \eta i_{js}^2 \mathrm{d}s\right)\Big| i_{jt} = i\right], \tag{10}$$

where q_t^j is shorthand notation for $\left(q_{jt}^a, q_{jt}^b\right)$ which contains the quantity sold (at the ask) and purchased (at the bid) by market maker j at time t, q^{-j} denotes the processes of all market makers *except* j, and η parameterizes the flow cost associated with non-zero inventory. η is a catchall parameter that, for example, increases in a security's fundamental volatility (σ), in market-maker risk-aversion, or in funding cost.

A Cournot equilibrium can now be defined as a strategy profile $q := (q^1, \ldots, q^N)$ where each q^j solves (10) taking q^{-j} as given. We focus on symmetric strategy profiles only (as is standard in the literature, e.g., Brunnermeier and Pedersen (2005)). Note that once equilibrium quantities are solved for, equilibrium prices follow immediately by plugging the quantities into (7).

Stackelberg game With all building blocks developed earlier in this section, we can now define equilibrium formally. It is a Stackelberg equilibrium where the large seller moves first and the market makers follow by taking the large-seller's action as input. Formally, the equilibrium is defined as follows.

Definition 1. *A Stackelberg equilibrium* (q^*, f^*) *consists of a profile of market maker (symmetric) strategies* q^* *and a trade intensity by the large seller* f^* *such that*

1. *Given* f^**, the strategy profile* q^* *is a Cournot equilibrium for the market maker game in the second stage as specified in* (10)*.*

2. *Given the strategy profile* q^* *and its implied price process,* f^* *maximizes the large seller's net proceeds as specified in* (5)*.*

In sum, the model features market makers who effectively provide a continuous quote stream, a large seller who trades at a pre-specified intensity and is constantly in the market for a fixed period of time, and other investors who arrive in the market at discrete (random) times. We believe that this setting captures the following salient features of a modern electronic market:

- Market makers can be interpreted to be high-frequency market makers (i.e., high-frequency traders who act as market makers). They invested in technology that enables them to interact with the market at high speed and to process trade information almost instantly.

- The large seller is unable to send his large (parent) order to the market at once. He therefore has to stay for a period of time and send a stream of small (child) orders to the market. He is likely to do so algorithmically. A popular algorithm is the volume-weighted average price

(VWAP) where one aims to trade at VWAP pertaining in the period during which one is trading. This effectively means trading in a fixed proportion to total volume throughout the interval. This feature matches the fixed sell intensity in the model if time is interpreted to run on a volume-clock (as opposed to a wall clock). The idea of securities markets running in operational time has long been in the literature and re-emerged recently (Clark 1973, Easley, López de Prado, and O'Hara 2012, Kyle and Obizhaeva 2016).

- Finally, other investors whose trade demand is moderate are likely to trade all they want upon arrival and then leave. We believe this lumpy nature of their trading matches real-world markets where investors with small orders arrive, trade, and leave. This lumpy nature could be relaxed in the model by taking an appropriate limit (i.e., take arrival rate to infinity and at the same time reduce their trade desire to zero). The resulting equilibrium is not necessarily more tractable but, more importantly, it loses the additional price volatility that lumpy arrivals generate.

2 Equilibrium

In this section we identify equilibria in the Stackelberg game. As the model is solved backwards we first discuss the second stage where market makers determine their optimal response to the large seller's action in the first stage. Given these results, we then discuss the first stage where the large seller strategically chooses his action *knowing* how market makers respond in the second stage.

2.1 Stackelberg second stage: Market-making equilibrium

2.1.1 Baseline case (no large seller)

Proposition 1 (Baseline case: No large seller)**.** *In the baseline case there is no large seller. There is a unique solution to the Stackelberg equilibrium as defined in Definition 1. The equilibrium quantities are time invariant. They only depend on a market maker's inventory state i. (Note that total inventory is Ni as all market makers are symmetric.) The equilibrium bid and ask prices are (expressed in terms of an individual market maker's inventory i):*

$$p^a(i) = A_\theta - B_\theta N i, \qquad \textit{(Ask price)} \qquad (11)$$

$$p^b(i) = -A_\theta - B_\theta N i, \qquad \textit{(Bid price)} \qquad (12)$$

and the bid-ask spread therefore is

$$s(i) = p^a(i) - p^b(i) = 2A_\theta, \qquad \textit{(Bid-ask spread)} \qquad (13)$$

Each market maker contributes the following quantity to the ask and bid price, respectively,

$$q^a(i) = C_\theta + D_\theta i, \qquad \textit{(Ask quantity)} \tag{14}$$

$$q^b(i) = C_\theta - D_\theta i. \qquad \textit{(Bid quantity)} \tag{15}$$

The value function for a market maker is

$$v^M(i) = E_\theta - F_\theta i^2. \qquad \textit{(Value market makers)} \tag{16}$$

where $A_\theta, \ldots, F_\theta > 0$ *are closed-form expressions in the deep parameters collected in* θ*.*

(All proofs of technical results are delegated to Appendix C.)

To generate economic insight the following corollary lets the number of market makers tend to infinity. It focuses on some key characteristics of liquidity. These include the bid-ask spread, total depth at the best bid and ask, and conditional price pressure. The latter is defined as the extent to which quotes get skewed per unit of aggregate market maker inventory Ni (Hendershott and Menkveld 2014). This aspect of liquidity is particularly important for large investors who, through their continued directional trading, push market makers into inventories and thus push the prices away from fundamental. For example, in the period that a large seller is executing the average position of market makers increases and quotes therefore become pressured downwards. This adds to the total execution cost for the large seller (i.e.., his implementation shortfall).

Corollary 1 (Full competition)**.** *The liquidity supplied by market makers yields relatively simple expressions for the fully competitive case (i.e.,* $N \to \infty$*). The bid-ask spread becomes*

$$\lim_{N\to\infty} p^a(i) - p^b(i) = \lim_{N\to\infty} 2A_\theta = 0, \tag{17}$$

the conditional price pressure is

$$\lim_{N\to\infty} B_\theta = 2\frac{\eta}{\beta}, \tag{18}$$

and the total *quantities offered become governed by*

$$\lim_{N\to\infty} NC_\theta = \delta\omega, \tag{19}$$

$$\lim_{N\to\infty} ND_\theta = 2\delta\frac{\eta}{\beta}. \tag{20}$$

If market makers start off with zero inventory (i.e., $i_0 = 0$*), then one obtains*

$$\lim_{N\to\infty} i_t = 0 \quad \forall t > 0, \tag{21}$$

$$\lim_{N\to\infty} v^M(0) = 0. \tag{22}$$

Table 1
Comparative statics bid-ask spread and conditional price pressure

This table summarizes how the bid-ask spread and the conditional price pressure change with the primitive parameters in the baseline equilibrium. These are based on signing partial derivatives (see the online appendix). The conditional price pressure is defined as the absolute value of midquote change per unit of total inventory change (Ni).

Parameter	Bid-ask spread	Conditional price pressure (B_θ)
Inventory-holding cost (η)	+	+
Investor arrival rate (λ)	–	–
Number of market makers (N)	–	–
Discount rate (β)	–	–
Price elasticity demand (δ)	+	–
Maximum private value investors (ω)	+	Unchanged

Corollary 1 yields several insights. First, the bid-ask spread disappears when infinitely many market makers enter. Second, this does not hold for price pressure as market makers continue to charge non-zero (conditional) price pressure (B_θ). Third, the sum of bid and ask depth ($2NC_\theta = 2\delta\omega$) reaches its maximum value.[11]

The expression for conditional price pressure B_θ yields some more insight. First, if holding inventory becomes more costly for market makers (i.e., higher η), then price pressure increases. This result is rather intuitive. Second, if market makers care less about the future (i.e., higher β) then price pressure declines. This finding is less trivial. The reason is that β affects the fundamental trade-off underlying the equilibrium price pressure. The trade-off is spending more money now to mean-revert out of non-zero inventory (i.e., higher price pressure) against staying longer on the inventory and suffer the pecuniary costs associated with it (captured by η). If the future is discounted more, then the trade-off is in favor of spending less now and therefore reducing price pressure. Note that this effect cushions liquidity deterioration at times of crisis when market makers much prefer a dollar now over a dollar tomorrow.

Table 1 provides comparative statics on the bid-ask spread and the conditional price pressure. It serves mostly as a sanity check. It signs the partial derivatives of two key illiquidity measures:

[11] This result follows from the following two observations. First, from (2) and (3) and the observation in footnote 10 it follows that total depth is:

$$q_t^a + q_t^b = \delta\left(2\omega - \left(p_t^a - p_t^b\right)\right). \tag{23}$$

Second, the bid-ask spread is constrained to be non-negative. It then immediately follows that the maximum value of the right-hand side of (23) is $2\delta\omega$.

the bid-ask spread and conditional price pressure. Table 1 indicates that both of these measures increase (i.e., the market becomes more illiquid) when the costs of holding the security are higher, fewer investors arrive, or fewer market makers supply liquidity.

The other results in Table 1 are less trivial. First, the market becomes more liquid when the discount rate increases. The reason is that the discount rate affects the trade-off between more money now and less disutility in the future. It is intuitive that market makers with a higher beta prefer to not pay as much money now for reverting to zero inventory and therefore apply lower (conditional) price pressure (see the discussion of Corollary 1 for detailed argument). They further seem to like the additional immediate revenue generated by more traffic due to a lower bid-ask spread and worry less about what this means for disutility in the future.

Second, a higher price elasticity of demand affects both measures of liquidity differently. It raises the spread but reduces conditional price pressure. The latter effect is intuitive, if net volume becomes more responsive to price, then market makers do not have to skew quotes as much. The bid-ask spread effect is more complex. Essentially, the increase in demand elasticity (i.e., higher δ) this gives market makers leeway to, for example, raise the ask price without changing ask volume, all else equal. As competition is soft, the equilibrium result finds that this is what market makers do. This effect disappears as we approach approaching a fully competitive market making sector (see Corollary 1).

Third, if investors arrive with higher values (i.e., higher ω) then the bid-ask spread increases and conditional price pressure remains unchanged. This result is not surprising after having discussed the elasticity effect. The spread again rises because market makers compete softly and manage to squeeze out additional rent from the larger demand surplus. The price pressure however remains unchanged since unlike elasticity, raising private values does not change net imbalance sensitivity to price skews. Raising ω adds an equal amount of securities from the buy and sell side, and therefore does not affect *net* imbalance.

2.1.2 Large seller, sunshine trading (☼)

Proposition 2 (Sunshine trading: Large seller)**.** *When there is a large seller who sells at constant intensity $f = f^{☼}$ from $t = 0$ until $t = d$ and this is common knowledge (i.e., sunshine trading) then the results developed in Proposition 1 change in the following ways. The equilibrium remains unique but the quantities are no longer time invariant. They depend on time (t) and market maker's inventory (i). (Note that total inventory is Ni as all market makers are symmetric.) The equilibrium bid and ask prices are (expressed in terms of an individual market maker's inventory i):*

$$p^{a☼}(i, f, t) = p^a(i) - 1_{\{t\le d\}} f \left(G^{☼}_{\theta}(d - t)\right), \qquad \textit{(Ask price)} \qquad (24)$$

$$p^{b☼}(i, f, t) = p^b(i) - 1_{\{t\le d\}} f \left(G^{☼}_{\theta}(d - t) + H^{☼}_{\theta}\right), \qquad \textit{(Bid price)} \qquad (25)$$

where $p^a(i)$ and $p^b(i)$ are the ask and bid quotes in the baseline case (Proposition 1). $G^{☼}_{\theta}(\tau)$ is zero for $\tau = 0$ and strictly increasing in τ, and $H^{☼}_{\theta}$ is a positive constant. Therefore, the bid-ask *spread is*

$$s^{☼}(i,f,t) = s(i) + 1_{\{t\le d\}} f H_\theta^{☼}, \qquad \textit{(Bid-ask spread)} \tag{26}$$

where $s(i)$ is the bid-ask spread in the baseline case. Each market maker contributes the following quantity to the ask and bid price, respectively,

$$q^{a☼}(i,f,t) = q^a(i) + 1_{\{t\le d\}} f \frac{\delta}{N}\left(G_\theta^{☼}(d-t)\right), \qquad \textit{(Ask quantity)} \tag{27}$$

$$q^{b☼}(i,f,t) = q^b(i) - 1_{\{t\le d\}} f \frac{\delta}{N}\left(G_\theta^{☼}(d-t) + H_\theta^{☼}\right). \qquad \textit{(Bid quantity)} \tag{28}$$

The value function for a market maker is

$$v^{M☼}(i,f,t) = v^M(i) + 1_{\{t\le d\}}\left(I_\theta^{☼}(d-t,f) + fJ_\theta^{☼}(d-t)\,i\right), \qquad \textit{(Val. mkt makers)} \tag{29}$$

where $I_\theta^{☼}(\tau,f) = 0$ for $\tau = 0$ or $f = 0$ and convex in f, and $J_\theta^{☼}(\tau) = 0$ for $\tau = 0$ and strictly decreases in τ.

Proposition 2 presents the equilibrium outcome in case a large seller arrives in the market at time zero, and sells at constant intensity f until d where the latter is common knowledge. We refer to this equilibrium as sunshine trading because d is known to all agents. It is useful to compare this equilibrium to the baseline one without a large seller. Such comparison yields the following insights which are all statements *conditional* on inventory being equal to i. First, the bid and ask prices are skewed downwards by an amount that is larger the further t is from duration d. The ask price smoothly converges to the corresponding value in the baseline case as t approaches d. The bid price, however, jumps up at time d by an amount equal to $H_\theta^{☼} > 0$. This sudden change in the bid price suggests that soft competition pulls non-zero rents from the large seller who consumes the bid while liquidating.

Second, the bid-ask spread is higher during the period the large seller is liquidating. The differential equals the amount by which the bid quote experiences additional pressure relative to the ask quote (i.e., $H_\theta^{☼}$).

Third, the depth at the bid and ask prices stays unchanged after d, i.e, after the large seller has left the market, but not before. During the liquidation period, there is more depth at the ask and less depth at the bid relative to the baseline case. This arguably reflects market makers' desire to sell more to end-user investors (at the ask) and buy less from them (at the bid) when the large seller is present. They anticipate that the large seller is selling in the time remaining until d, and therefore compensate by accommodating positive net volume now.

Fourth, all else equal, the presence of a large seller has a non-trivial effect on a market maker's value. The difference in value function between the baseline and sunshine case consists of two terms: an (inventory) state-dependent term ($fJ_\theta^{☼}(d-t)\,i$) and a constant term $I_\theta^{☼}(d-t,f)$. The state-dependent term captures the benefit of being short as a market maker while the large seller is executing (i.e., $J_\theta^{\star}(\tau) < 0$ for $\tau > 0$). In other words, it is a good state to be in when absent any price pressure: the net future flow is expected to be negative and market makers are therefore expected to be net buyers[12], more so the larger the net selling pressure is (i.e., f) and the longer it lasts (i.e., τ). This benefit of being short equals the cost of being long in this state (i.e., the term is of equal magnitude but carries the opposite sign when considering a particular long instead of a short position).

The state-independent term $I_\theta^{\star}(d-t, f)$ equals the value differential absent any inventory. This expression therefore captures whether market makers who start with a zero inventory are better off with or without the large seller (i.e., $I_\theta^{\star}(d-t, f)$ positive or negative, respectively). It turns out that the sign of $I_\theta^{\star}(d-t, f)$ depends on duration d and a "risk-capacity condition," as summarized in the following lemma.

Lemma 1 (Market maker value sunshine trading)**.** *Market makers benefit from the presence of a large seller for any given large-seller liquidation intensity (f) if*

- *either the risk-bearing capacity of the market is sufficiently high (i.e, condition* (48) *in Appendix* C *holds which is the case if the inventory-holding cost (η) is low enough, or there are sufficiently many market makers (N), or if the investor arrival rate (λ) is high enough),*
- *or the duration of the large-seller's presence is short.*

Lemma 1 suggests that market makers benefit from the large seller's presence unless conditions are truly adversarial: low risk-bearing capacity and a long-lasting sell order. The exact conditions are somewhat involved and have therefore been relegated to the proof of the lemma in the appendix. The analysis thus far suggests the following trade-off. The large-seller's presence guarantees sell traffic which enables the market makers to generate some rents off of bid side by skewing the bid more than they otherwise would have done (i.e., the $H_\theta^{\star}$ term in (25)). The cost of this traffic is that they need to hold on costly inventory for longer, in particular when the directionality in the order flow lasts for a long time. This explains why for low enough risk-bearing capacity and long enough duration market makers are worse off when there is a large seller.

Finally, the equilibrium yields tractable expressions for the expected path of the market maker inventory and, relatedly, the expected bid and ask price paths. The following lemmas capture the insights that these expressions yield.

Lemma 2 (Expected market maker inventory path)**.** *The expected inventory of market makers*

- *strictly increases in time while the large seller is executing,*
- *is homogeneous of degree one in the large seller's trade intensity,*
- *exhibits exponential decay towards zero after the seller disappears.*

[12] Absent any price pressure is defined as quotes centered symmetrically around fundamental value.

Lemma 2 shows that market maker inventory gradually increases while the large seller executes and decays to zero exponentially after the large seller is done executing. Interestingly, if the large seller doubles his intensity then expected inventories are twice as high. These expected inventory paths are then used to characterize prices that are linear in inventory resulting in the following lemma.

Lemma 3 (Expected bid and ask price paths)**.** *The expected bid and ask price paths for $t \leq d$*

- *jump down at $t = 0$ (relative to the steady state in the baseline case),*
- *are convex in t,*
- *strictly decline initially,*
- *and potentially increase in t provided d is large enough.*

For $t > d$ the price paths

- *are concave in t and*
- *strictly increase in t.*

Lemma 3 characterizes expected price paths and shows that prices start off being pressured downwards and become more pressured at least initially. They, however, might start to show less pressure *before* the large seller leaves the market. Such easing of price pressure is in line with the empirical evidence provided by Zarinelli et al. (2015, Fig. 8). If the large seller is about to stop selling, then market makers reduce the subsidy they effectively hand out to end-user investors by applying price pressure.

2.1.3 Large seller, stealth trading (☁)

Proposition 3 (Stealth trading: Large seller)**.** *When there is a large seller who sells at constant intensity $f = f^{☁}$ from $t = 0$ until $t = d$ where d is unknown to market makers (i.e., stealth trading) then the results developed in Proposition 1 change in the following ways. The equilibrium remains unique with bid and ask prices for each market maker equal to*

$$p^{a☁}(i, f, t) = p^{a}(i) - 1_{\{t \leq d\}} f \left(G_{\theta}^{☁}\right), \qquad \textit{(Ask price)} \tag{30}$$

$$p^{b☁}(i, f, t) = p^{b}(i) - 1_{\{t \leq d\}} f \left(G_{\theta}^{☁} + H_{\theta}^{☼}\right), \qquad \textit{(Bid price)} \tag{31}$$

where $p^a(i)$ and $p^b(i)$ are the ask and bid quotes in the baseline case (Proposition 1) and $H_\theta^{\text{☼}}$ is a positive constant (featured also in Proposition 2). The associated quantities are

$$q^{a\text{☁}}(i,f,t) = q^a(i) + 1_{\{t\le d\}} f \frac{\delta}{N}\left(G_\theta^{\text{☁}}\right), \qquad \textit{(Ask quantity)} \qquad (32)$$

$$q^{b\text{☁}}(i,f,t) = q^b(i) - 1_{\{t\le d\}} f \frac{\delta}{N}\left(G_\theta^{\text{☁}} + H_\theta^{\text{☼}}\right), \qquad \textit{(Bid quantity)} \qquad (33)$$

and the bid-ask spread therefore is

$$s^{\text{☁}}(i,f,t) = s(i) + 1_{\{t\le d\}} f\, H_\theta^{\text{☼}} = s^{\text{☼}}(i,f,t). \qquad \textit{(Bid-ask spread)} \qquad (34)$$

The value function for a market maker is

$$v^{M\text{☁}}(i,f,t) = v^M(i) + 1_{\{t\le d\}}\left(I_\theta^{\text{☁}}(f) + f J_\theta^{\text{☁}} i\right), \qquad \textit{(Value market makers)} \qquad (35)$$

where $I_\theta^{\text{☁}}(f) = 0$ for $f = 0$ and convex in f, and $J_\theta^{\text{☁}} < 0$.

The stealth equilibrium characterized in Proposition 3 is very similar to the sunshine equilibrium of Proposition 2. The differences generate insight in what opacity does to the market. First, note that all quantities become time invariant as in the baseline equilibrium. This is required as it guarantees that market makers are unable to learn the duration of the large seller's order. At time t the future is either unchanged relative to an instant ago ($t - \mathrm{d}t$) if the large seller is still trading, or it switches to the baseline-case future if the large seller just stopped trading.

Second, the bid and ask prices both jump to the corresponding quantities in the baseline case at the moment the large seller stops trading. In the sunshine equilibrium the transitions are smooth as the G term decays to zero with time progressing towards d (cf. (24) and (25)). In the stealth equilibrium, instead, the term G is a constant which causes both prices to jump.

Third, similar to sunshine, market makers bid conservatively during the liquidation period and this causes an extra skew to the bid relative to the ask. As discussed in Section 2.1.2, we believe this effect reflects market power exercised by market makers in the presence of guaranteed selling. Interestingly, the size of the extra skew is equal across both equilibria ($H_\theta^{\text{☼}}$). Coincidentally, this is why in sunshine unlike the ask price, the bid price jumps at the time the seller disappears from the market.

Fourth, the bid-ask spread in stealth equals the spread in sunshine. In both equilibria the spread is elevated during the liquidation period. The increase is entirely driven by the additional skew in bid prices. The differential in equilibrium prices across stealth and sunshine is therefore entirely driven by the price pressure captured by G, not by the spread.

Fifth, comparing market maker value in the stealth case with the baseline case of no large seller yields similar insights as in the sunshine versus baseline case. The only salient difference is that the time that remains until the large seller leaves is no longer a state variable for the simple reason that a market maker does not know it in stealth trading. For example, the coefficient $J_\theta^{\text{☁}}$ in the value differential across stealth and baseline does not depend on $(d-t)$. It however varies with other parameters in the same way as $J_\theta^{☼}(d-t)$ does. The same observation holds for $I_\theta^{\circlearrowright}$ and the equivalent of Lemma 1 for stealth only features the risk-capacity condition. More specifically, the presence of a large seller benefits a market maker only when risk-bearing capacity is large enough (see Lemma 8 in Appendix B).

The broader question is whether market makers prefer stealth over sunshine. To judge whether they do, one would compare the value functions in the two trading regimes, given respectively in (35) and (29). Unfortunately, the differential is intractable. We will revisit this question after endogenizing the large-seller's trading. It turns out that then the differential becomes tractable in the case of full market making competition (i.e., $N \to \infty$).[13]

Just as in sunshine, the expected inventory path can be characterized analytically. The result is similar and stated as Lemma 9 in the appendix. The more interesting result is on expected price paths which differ across sunshine and stealth. Before discussing the difference we state the result for stealth in the following lemma.

Lemma 4 (Expected bid and ask price paths)**.** *The expected bid and ask price paths for $t \leq d$*

- *jump down at $t = 0$,*
- *are convex in t, and*
- *strictly decline in t.*

For $t > d$ the price paths

- *are concave in t and*
- *strictly increase in t.*

Lemma 4 characterizes expected price paths and shows that prices start off being pressured downwards and become more pressured during the liquidation period. Note that contrary to sunshine trading, price pressure never subsides before the large seller leaves the market. With stealth trading market makers are unable to forecast when the seller leaves the market and it is therefore optimal for them not to relinquish pressure. Not having seen the large seller leave in the last instant leaves the market makers' assessment of future probabilities of the large seller disappearing unchanged. This finding therefore sets stealth trading apart from sunshine trading. The evidence cited in Section 2.1.2 of pressure subsiding before the large seller leaves the market therefore suggests sunshine trading is the more likely equilibrium pertaining in real-world markets.

[13] The result is stated as Lemma 6.

2.2 Stackelberg first stage: Strategic large seller

Having solved the second stage of the game, we now turn to the first stage. The large seller's objective is to maximize the proceeds of selling net of his reservation value as specified in (4). He understands what market makers' optimal response is to any level of sell intensity he considers. This response drives the (endogenous) bid price process (b_t). In the case of sunshine trading, he sets the intensity such that net proceeds are maximized:

$$f^{\text{☼}}(d) = \arg\max_{f} v^{S\text{☼}}(f) := \arg\max_{f} \mathbb{E}\left[P_d(f)|d \text{ revealed}\right]. \tag{36}$$

In the case of stealth trading the large seller cannot condition his sell intensity on d (as otherwise market makers learn). Hence, he takes the expectation with respect to d and picks the intensity as follows:

$$f^{\text{☁}} = \arg\max_{f} v^{S\text{☁}}(f) := \arg\max_{f} \int_0^\infty \mathbb{E}\left[P_s(f)\right] \nu e^{-\nu s} ds. \tag{37}$$

The optimal intensities are derived in closed-form. The exact expressions appear in the proof of the following lemma that characterizes them.

Lemma 5 (Optimal liquidation intensity)**.** *Let $f^{\text{☼}}(d)$ and $f^{\text{☁}}$ denote the optimal liquidation intensity of the large seller in the first stage of the Stackelberg game, then*

- *$f^{\text{☼}}(d)$ is continuous and strictly decreasing in d and*
- *the stealth intensity is wedged between the lowest and highest sunshine intensity:*

$$0 < \lim_{d\to\infty} f^{\text{☼}}(d) < f^{\text{☁}} < f^{\text{☼}}(0) = \frac{N\delta\lambda\omega}{2} < \infty. \tag{38}$$

With the optimal intensities, we can now compare the net proceeds across sunshine and stealth trading. This leads to this section's signature result stated in the following proposition.

Proposition 4 (Stealth or sunshine)**.** *The expected proceeds for the large seller are strictly larger for the sunshine equilibrium as compared to the stealth equilibrium. This is not due to the large seller's ability to condition his optimal sell intensity on d as the inequality continues to hold if he is forced to trade at the optimal stealth intensity for all d. Formally, this result is described as*

$$\underbrace{\mathbb{E}\left[P_d\left(f^{\text{☼}}(d)\right)|d \textit{ revealed}\right]}_{\textit{Net proceeds sunshine}} > \mathbb{E}\left[P_d\left(f^{\text{☁}}\right)|d \textit{ revealed}\right] > \underbrace{\mathbb{E}\left[P_d\left(f^{\text{☁}}\right)|d \textit{ hidden}\right]}_{\textit{Net proceeds stealth}}, \tag{39}$$

where P denotes the net proceeds for the large seller (i.e., net of his reservation value).

Proposition 4 finds that the large seller is strictly better off when the duration of his trading is revealed. This is not solely due to his ability to condition the sell intensity on duration. The reason is that even if he picks the optimal stealth intensity, then his net proceeds are higher if duration is revealed. The differentials however decline to zero when the number of market makers is taken to infinity as stated in Lemma 6.

The surprising and powerful part of the proposition's result is that the *large seller* benefits from revelation of d. That such information benefits a market maker is intuitive as it enables him to fine-tune his price policy. The non-trivial result is that the large seller benefits as well. This does not rely on market makers passing on the benefits due to competition *per se*. This result persists even in the monopolistic case (i.e., $N = 1$).[14]

Lemma 6 (Sunshine vs. stealth on full competition)**.** *If the number of market makers is taken to infinity, then*

- *in the baseline case the value function of a single market maker converges to*

$$\lim_{N\to\infty} v^{M}(i) = -\frac{\eta}{\beta} i^2, \tag{40}$$

- *in stealth it converges to*

$$\lim_{N\to\infty} v^{M\text{☁}}(i, f^{\text{☁}}, t) = -\frac{\eta}{\beta} i^2 + \frac{1}{\delta\lambda(\nu+\beta)} \left(\frac{\omega}{4\frac{\eta}{\beta(\nu+\beta)} + \frac{2}{\delta\lambda}} \right)^2 1_{\{t<d\}}, \tag{41}$$

- *and for sunshine we have*

$$\lim_{N\to\infty} v^{M\text{☼}}\left(i, f^{\text{☼}}(d), t\right) = -\frac{\eta}{\beta} i^2 + \frac{1}{\delta\lambda} \left(\frac{\omega}{4\frac{\eta}{\beta^2}(1 - \frac{\beta d}{e^{\beta d}-1}) + \frac{2}{\delta\lambda}} \right)^2 \frac{1 - e^{-\beta(d-t)}}{\beta} 1_{\{t<d\}}. \tag{42}$$

Note that in the baseline case, market maker value for a zero inventory position converges to zero whereas it remains strictly positive for sunshine and stealth.

Lemma 6 finds that sunshine strictly dominates stealth for market makers even under full competition (i.e., $N \to \infty$). This result is based on a comparison of their value function on a zero inventory position, where for sunshine the large-seller order duration is integrated out. The lemma further shows that, under full competition, market maker value (again assuming to start with zero inventory) is zero in the baseline case of no large seller. If there is such large seller, then irrespective of sunshine or stealth trading, market maker value remains strictly positive under full competition. This result is surprising and illustrates the importance of having endogenized both sides of liquidity as stated in the following corollary:

[14]Coincidentally, this effect explains why a client willingly discloses how long his trade horizon is when negotiating a deal with a sell-side broker who would act as a market maker accommodating his flow. Such transaction is referred to as a "principal bid transaction."

Corollary 2 (Endogenous liquidity demand has bite)**.** *Liquidity-demand endogeneity is the root cause for market makers earning positive rents under full competition in the presence of a large seller (see Lemma 6). That is, this result vanishes when the execution intensity of the large seller is fixed at an exogenous level, say $f^e(d)$ (now independent of N). We then fall back to the baseline result of no rents for a single market maker under full competition (see also (40)):*

$$\lim_{N\to\infty} v^{Me}\left(i, f^e(d), t\right) = -\frac{\eta}{\beta} i^2. \tag{43}$$

The intuition for the corollary's result is that the large seller raises his execution intensity (to infinity) when the number of market makers is taken to infinity. This concurrent demand-increase creates space for liquidity suppliers to retain positive rents.

Finally, we establish the intuitive result that the large-seller's participation rate decreases in d for the sunshine case. This is to be expected as his optimal sell intensity decreases in d. It nevertheless requires work to be shown because investor volume is endogenous and depends on market makers' optimal response to the liquidation intensity under sunshine trading. Despite the expected participation rate cannot be solved in closed-form, we can solve explicitly for the expected *reciprocal* of the large-seller's participation rate. We state the result in the following lemma.

Lemma 7 (Reciprocal of large-seller participation rate)**.** *The expected reciprocal of the large-seller participation rate in the interval he trades in strictly increases with d and is given by:*

$$\mathbb{E}\left(\frac{Volume}{d \times f^{\star}(d)} | d\right) = K_\theta + L_\theta \frac{1}{f^{\star}(d)}, \tag{44}$$

where "Volume" is total volume from $t = 0$ until $t = d$ (including both large-seller and investor trades with market makers).

Lemma 7 finds that the expected reciprocal of the large-seller's participation rate strictly increases in d. This suggests that the expected large-seller's participation rate decreases in d. This is in line with the empirical evidence on large-order executions (Zarinelli et al. 2015, Fig. 2). More importantly, it makes the sunshine equilibrium easier to implement in real-world markets. The large-seller does not need to reveal his duration as market makers infer it from the strength of directionality in order flow. This lends further credibility to the equilibrium as explaining empirical patterns (because large sellers do not literally announce d to all market makers).

3 Welfare

In this section we study the welfare effects of the large seller's presence and his decisions. The effect of the seller on market makers was already analyzed in Proposition 2 and the corresponding discussion. Next, it remains to analyze how it affects other investors. Section 3.1 presents such analysis for both sunshine and stealth trading. To assess the (social) inefficiency of sunshine trading, Section 3.2 compares its outcome to the corresponding outcome when a social planner sets prices.

3.1 Large-seller's effect on end-user investors

The next proposition describes under what conditions the presence of a large seller benefits end-user investors. Such assessment requires defining what the value of trading is to investors. A natural definition is to set it equal to the expected realized surplus of all future investors, discounted appropriately, assuming market makers start off on a zero inventory. Mathematically, this definition is:

$$v^I = \mathbb{E}\left[\int_0^\infty e^{-\beta s}\left(\frac{1}{2}\delta\left(\omega - p_s^a\right)^2 \mathrm{d}N_s^B + \frac{1}{2}\delta\left(\omega + p_s^b\right)^2 \mathrm{d}N_s^S\right)|i_{j0} = 0\right]. \tag{45}$$

Proposition 5 (Large-seller's effect on end-user investors)**.** *The presence of a large seller has a non-trivial effect on end-user investors. For both sunshine and stealth trading realized end-user investor surplus is convex in f. In particular, there is a threshold $\bar{f}$ such that*

- *end-user investors are strictly worse for $f < \bar{f}$,*
- *equally well off when $f = \bar{f}$, and*
- *strictly better off when $f > \bar{f}$.*

Proposition 5 summarizes how the presence of a large seller affects end-user investors. It turns out that the effect depends on how aggressively the large seller executes. For low sell intensities investors are worse off whereas for high intensities they are better off. This is due to the interplay of two effects. Proposition 2 and 3 show that for both sunshine and stealth, respectively, the presence of a large seller raises the spread and increases price pressure. The former results hurts investors whereas the latter benefits them. Market makers skewing quotes essentially means that they hand out a subsidy to arriving end-user investors to mean-revert their inventory.

The intuition for investors only benefiting from the large seller's presence in case the seller trades at high enough intensity is based on a trade-off. The seller's presence causes the bid-ask spread to be larger but pressures prices downwards relative to fundamental value. The spread effect hurts investors, the pressure effect benefits them as they are net buyers in this period.

How does the size of the two effects depend on the large seller's trade intensity? The increase in the bid-ask spread scales linearly with the seller's trade intensity by the same amount in sunshine and stealth (i.e., by $H_\theta^{☼} f$, see Proposition 2 and 3). The conditional price pressure is unaffected by the large seller's presence and amounts to $B_\theta N$ in all three cases: baseline, sunshine, and stealth (see Proposition 1 through 3). Yet, more intense selling implies that market makers hold more inventory which causes prices to become pressured more. This effect is super-linear given that increasing the sell intensity by a unit per dt adds price pressure contemporaneously, but also in the future given that inventory needs to mean-revert. The spread cost being linear in the seller's intensity and the pressure-benefit being super-linear explains that there is a threshold seller intensity beyond which investors benefit from the large seller's presence.

3.2 Social inefficiency

To judge social inefficiency we study a constrained first-best where a social planner sets bid and ask prices. This case, we believe, is most interesting as a regulator has some control over prices[15], but cannot control trade decisions of investors. More specifically, this constrained first-best features a planner who sums the discounted net proceeds of the large seller (see (4)), the discounted gross earnings minus inventory cost of the market makers (see (10)), and the discounted realized surplus of investors (see (45)). The planner maximizes this welfare over the set of price policies that he could implement in the second stage of the Stackelberg game. Note that, although the planner only *controls* prices in the second stage, for each price policy he considers he needs to solve the large seller's optimal response in the first-stage to compute the implied welfare.[16]

Note that we rule out a trivial solution where the planner effectively also sets the large seller's liquidation intensity.

The planner problem can be solved analytically and the following proposition summarizes the results when compared to the sunshine and stealth equilibrium described in Proposition 2 and 3, respectively.

Proposition 6 (Social-planner solution)**.** *If instead of the market makers, a social planner sets prices then the Stackelberg equilibrium changes in the following ways both compared to sunshine and to stealth:*

- *The bid-ask spread is strictly lower.*
- *The spread is no longer elevated in the liquidation period. It becomes a liquidation-invariant constant.*
- *In the sunshine case, the large-seller trades at higher intensity for short durations. In fact, rather than being finite the limit intensity is infinite when the seller's horizon d is taken to zero:*

$$\lim_{d \downarrow 0} f^{☼ Planner}(d) = \infty. \tag{46}$$

[15]For example, some markets feature circuit breakers to dampen sudden strong price changes.

[16]An alternative social-planner setup would have the planner control both prices *and* the large seller liquidation intensity. The planner could even implement such solution with only control over prices in the second stage. That is, he could threaten to set the bid at large seller's reservation value (i.e., $-\omega$) if the seller deviates from the intensity that the planner deems optimal. Such threat however would be time-inconsistent as the planner would change his mind when the seller picks an intensity different from what the planner prescribes (and commits to it).

Proposition 6 yields several insights. First, the planner's solution exhibits a lower bid-ask spread. This demonstrates that market makers exercise market power when setting bid and ask prices. Second, the elevated spread in the liquidation period due to an additional skew on the bid price is gone in the planner's equilibrium. As discussed earlier, this shows that the skew is again due to market maker power at a time when the large seller is present in the market. Third, the large seller optimally sheds more of his position in the sunshine case, at least for short durations. The externality that arguably drives this result is that market makers apply too little price pressure and therefore stay on high inventory levels for too long. The planner internalizes the benefit that such price pressure generates for end-user investors as they pocket the subsidy. The planner will therefore happily accommodate more of the large-seller flow at least in the short term. It is simply less costly for the planner to mean-revert out of it.

4 Calibration

4.1 Calibration procedure

The calibration procedure picks the seven model parameters to match equally many real-world variables for the baseline case where time is measured in days:

- $\beta = -\log(0.9998)$: In their discrete-time model, Hendershott and Menkveld (2014) set the daily discount rate equal to 0.9998. The continuous-time analogue is the negative of the log.
- $\nu = 5$: The mean duration of large orders according to Zarinelli et al. (2015) is approximately 0.2 days. The mean of the exponential distribution is $1/\nu$ and we therefore set ν equal to five.
- $N = 10$: We believe a natural real-world equivalent of the model's market makers are high-frequency traders. Therefore the number of market makers was set to ten based on Scandinavian equity trading on Nasdaq-OMX.[17]
- The remaining four parameters are based on Menkveld (2013) who analyzed trading by one large high-frequency market maker for a 2007-2008 sample of Dutch blue chip equities. Its particular appeal is that it contains estimates of the conditional price pressure.[18] The calibration focuses on afternoon trading in large-cap stocks.

[17] "Europe's Top 10 High-Frequency Kingmakers (in Scandinavia, at Least)," Financial News, 2014.

[18] In this period, we believe it is reasonable to assume that there were ten market makers in the model (see earlier point). Unfortunately, Menkveld (2013) presents detailed trade results for only one of them who happened to be (temporarily) large in the sample. We interpret this HFT as being under strong competitive threat. We caution that the calibration only serves to find "reasonable" parameters. Estimation of the model is beyond reach as it requires detailed HFT data that is not available to us. We leave that for future research.

– $\lambda = 791$: The arrival rate of end-user investors needs to match the observed 1582 trades per day (Menkveld 2013, Table 1). The model-implied expected arrivals per day is 2λ (i.e., buyers and sellers added up).

– $\eta = 0.14$ bps/€1000, $\delta =$ €11,289/bps, and $\omega = 13.6$ bps where "bps" is basis points: These parameters are set at values to match the following three variables: Bid-ask spread (3.2 bps), conditional price pressure (0.026 bps/€1000), and the standard deviation of inventory (€80,300) (Menkveld 2013, Table 1 and 7). The three parameter solve a system of three equations with three unknowns which has a unique solution.

4.2 Calibration results

This subsection uses the calibrated model to illustrate various properties of the equilibria analyzed thus far. The calibration further allows us to get a sense of the economic magnitude of various effects for a sample of plain-vanilla actively traded stocks. The various subsections focus on the sunshine equilibrium that we believe best matches real-world trading of large sellers. As first discussed in the introduction, it matches two empirical properties that the stealth equilibrium cannot match: a negative correlation of the seller's trade intensity and his duration and the price pressure subsiding before a large seller finished trading (suggesting that market makers have some ability to predict the seller's trading is about to end). The subsections compare sunshine to stealth (4.2.1), to baseline (without large seller, 4.2.2), and to constrained first-best (4.2.3).

4.2.1 Sunshine versus stealth

Figure 2 illustrates the large seller's optimal sell intensity in the Stackelberg game for both sunshine and stealth trading. The top graph plots the sell intensity as a function of trade duration. This duration is exponentially distributed with a mean equal to one fifth of a trading day (i.e., $\nu = 5$). The top plots leads to a couple of observations. First, in the case of stealth trading the large seller cannot condition on his duration draw (to sustain opaqueness) and optimally trades at a duration-invariant intensity of €105 million per day.

Second, in sunshine the seller tends to trade at a much higher intensity in particular for relatively short orders. For example, orders that last less than 5% of a trading day which is approximately $5\% \times 8 \text{ hours} = 24 \text{ minutes}$ trade at least at 50% higher intensity. This is a non-trivial set of orders as duration draws less than 5% carry a probability of $\left(1 - e^{-5\times 0.05}\right) = 0.22$. The sell intensity grows to its upper bound of $N\delta\lambda\omega/2 = 10 \times \text{€}11,289\text{ /bps} \times 791\text{ /day} \times 13.6\text{ bps} =$ €1.21 billion/day for vanishingly small duration draws (see Lemma 5).

Third, for orders longer than 5% the sell intensities in sunshine and stealth are of equal magnitude. The sunshine intensity is slightly higher for durations shorter than 60%, and slightly lower for longer durations.

The bottom plot of Figure 2 illustrates the additional net proceeds the large seller gets in sunshine relative to stealth. The differential is hump shaped and peaks at a approximately 2.5% dura-

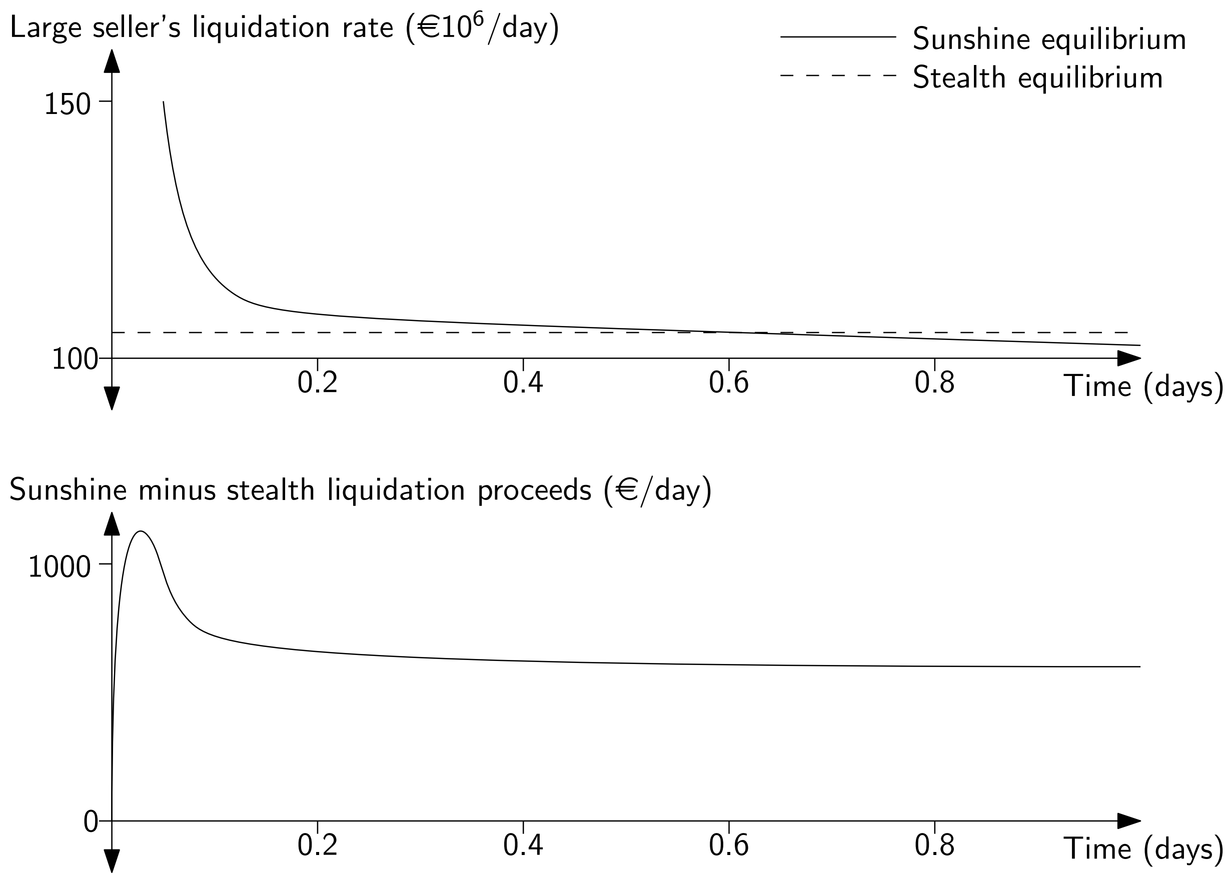

Figure 2. Calibrated model visualization. This figure illustrates the large seller's optimal trading strategy in the calibrated model where the market makers start off on a zero inventory. The top graphs shows, both for the sunshine and stealth case, what the optimal sell intensity is as a function of the seller's trade duration. The bottom graph plots the differential in expected discounted net liquidation proceeds across the two cases (i.e., sunshine minus stealth).

tion (where $P(d < 2.5\%) = 0.12$). At that point the discounted expected net proceeds are €1,100 higher in sunshine. For larger durations this differential drops and seems to asymptote to €800.

The expected benefit of sunshine over stealth is 5.5%. This expectation involves integrating out d in the expression for discounted net proceeds. 4.2% of this 5.5% benefit is due to switching to sunshine trading *per se* (i.e., keeping the liquidation intensity fixed). The remaining 1.3% is due to the ability to condition selling intensity on duration.

Further calculation reveals that the sunshine not only benefits the large seller, it also benefits market makers and end-user investors. The value increase for them are smaller in absolute terms than what it is for the large seller, but larger in relative terms. Market maker value's increase relative to the baseline is 7.6% higher for sunshine and investor value's increase relative to the baseline increases is 11.5% higher for sunshine.

Figure 3 characterizes trading when the large seller draws $d = 0.2$ (i.e., the mean duration). It mirrors Figure 1 but rather than a single realization, it plots the expected quantities given the calibrated parameters. In the top graph, for example, discrete end-user arrivals become smooth end-investor trade intensities based on their expected arrival rate. In all graphs, the solid lines illustrate the sunshine equilibrium whereas the dashed lines illustrate the corresponding stealth trade equilibrium.

The figure merits a few comments. First, the top graph illustrates that, for this average duration, the large seller trades more intensively in sunshine. This underlines one of the paper's key messages that transparency benefits the large seller which enables him to sell more (see Proposition 4). Note however that the illustration is for a mean draw of duration and therefore pertains to "average trading." For extremely large duration, the large seller trades less intensively which a stealth seller cannot do without revealing his duration (see discussion following Proposition 3).

Second, perhaps the most salient feature of the figure is the wedge between the inventory market makers collectively take on in the sunshine and the stealth equilibrium. In the case of stealth trading, inventories grow explosively right from the start and seem to asymptotically approximate €60,000. For sunshine, instead, inventories grow slowly towards approximately €200,000.

Clearly, market makers in stealth banked on a left-tail draw when quickly taking on inventory. This is expected given that the exponential distribution is left skewed. However, by the time 10% of the trading day has passed, market makers learned that it was not a short-duration order and effectively stop taking on directional flow. Under sunshine, instead, they continue to take on inventory, notably at progressively faster rate, until the order ends at 20% of the trading day. The difference is significant as, by the time the large seller stops trading, market makers have taken on an approximately *three times* larger position and thus effectively supplied more liquidity to the large seller. This result illustrates that duration-opacity can cost the large seller dearly.

Third, echoing the previous point, the price patterns in the bottom graph illustrate that, for a mean duration draw, initially "stealth market makers" skew quotes less than sunshine market makers but eventually skew quotes more. The top graph shows that, as a result, the expected volume traded by buyers is lower initially but higher eventually when comparing stealth to sunshine. End-user seller's expected traded volume shows the opposite pattern, i.e., higher initially and lower

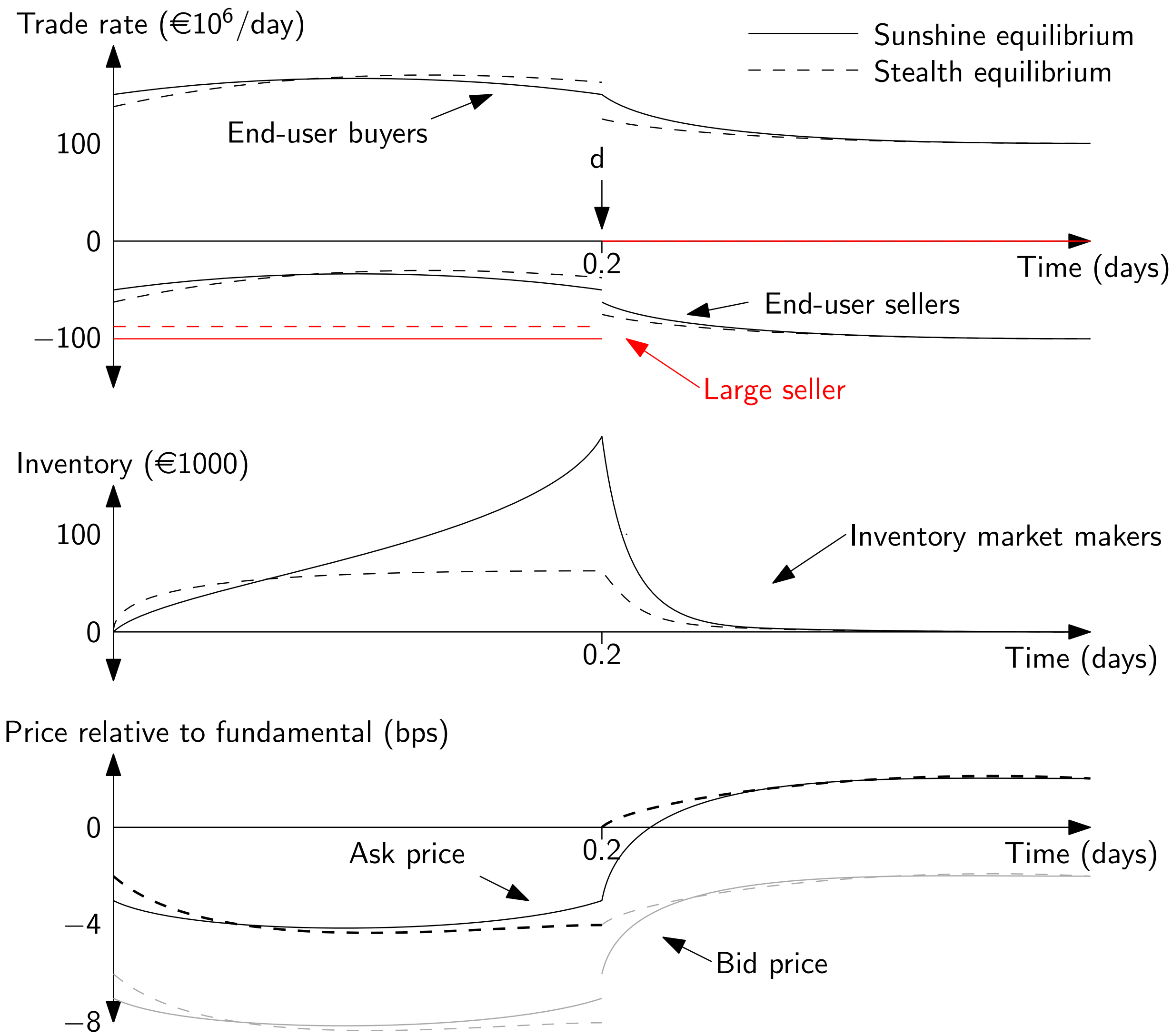

Figure 3. Calibrated model visualization. This figure illustrates the model by plotting the mean of various model variables based on the calibration. The top graph plots the (continuous) net flow of the seller as well as the mean arrival rate of buyers and sellers. The middle graph plots market makers' inventory which result from the net flows in the top graph (assuming they start off at zero inventory). The bottom graph plots the bid and ask prices relative to fundamental value.

eventually. The *net* end-user expected buy volume therefore is lower initially for stealth but higher eventually. This explains the market maker inventory patterns discussed in the previous point.

Fourth, the figure illustrates that for these calibrated parameters, the equilibrium prices render an aggressive preying strategy unprofitable. A natural candidate would be to sell a unit initially along with the large seller and turn around eventually to buy a unit at depressed prices. The midquote does slide initially, suggesting that such a strategy might be a worthwhile deviation strategy for a single market maker. Two features, however, make the strategy unprofitable. First, prices jump down at the moment the large seller starts executing. It is therefore impossible to sell at the baseline-case bid that prevailed just before the large seller starts executing. Second, the size of the bid-ask spread dominates the slide in midquote. That is, the revenue generated from selling at the bid initially is smaller than the cost of buying at the ask eventually. In other words, the lowest ask is higher than the highest bid throughout the large-seller's trading period.

4.2.2 Sunshine versus baseline

How the presence of a large seller benefits the various types of agents can be learned from comparing the sunshine case to the baseline case that does not feature a large seller. Such comparison leads to several observations. First, the seller's presence benefits both market makers and end-user investors but the effect is negligible. A market maker's value increase is 0.40 basis point whereas the investors' value increases by 0.05 basis points.

Comparative statics analysis shows that this result is robust. For investors, either doubling the number of market makers or reducing them to a single one, or changing inventory-holding cost by 10,000 does not change the result (i.e., investors remain better off in the presence of a large seller). For market makers, the result is similar except when for a reduction of the number of market makers to two or a reduction in inventory-holding cost by 10,000 makes them worse off (they are still better off in case of a reduction of the number of market makers to three and an inventory-holding cost reduction of 1,000).

4.2.3 Sunshine versus constrained first-best

Finally, comparing the sunshine equilibrium to the constrained first-best where the planner decides prices yields the following insights. First, the social inefficiency is economically small: 1.4%. Comparative statics show that this hinges critically on the assumption of there being ten market makers. In the case of a monopolistic market maker the inefficiency is 24.4%. For a duopoly the inefficiency is 11.1%.

Second, Figure 4 illustrates the large seller's optimal trade intensity for sunshine and (constrained) first-best. The graphs show that the intensity is substantially higher in the first-best outcome for all likely levels of duration. The difference is at least a third throughout. First-best therefore yields substantially more re-allocation of securities away from the large seller. If one integrates out duration, then we find the first-best intensity to be 372% higher.

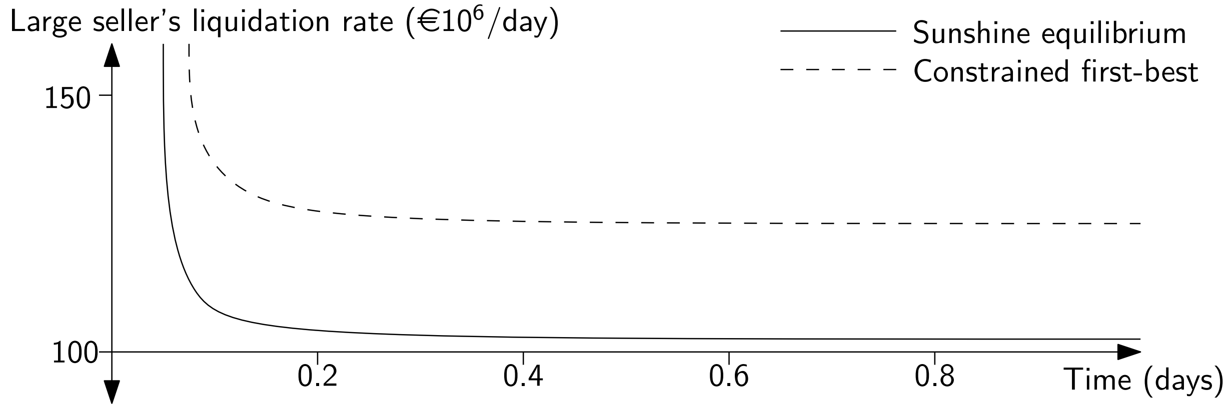

Figure 4. The optimal sunshine liquidation intensities in sunshine (solid) and constrained first-best (dashed). The intensities are based on the calibrated model where market makers start off on a zero inventory.

Third, the bid-ask spread becomes negligible in first-best as compared to sunshine. In the latter case it is 4.5 basis points during liquidation and drops to 3.2 basis points once the seller leaves. In first-best, the spread is only 0.03 basis points.

5 Conclusion

We propose a setting that enables us to integrate two types of literatures, one focused on endogenizing liquidity supply and the other doing so for liquidity demand. We propose a Stackelberg structure where the demander of liquidity is a single large seller who decides in the first stage at what intensity to sell, unconstrained by the total quantity but constrained by the time he has to execute the order. The seller takes into account how a set of market makers who are Cournot competitors responds to his order.

A particularly appealing feature of our setting is that it is possible to solve for a unique symmetric equilibrium in closed-form. This allows to thoroughly analyze various important properties of market making, price pressures, and optimal execution. In this paper we focus on analyzing whether the larger seller is better off in a transparent environment where duration is revealed or in an opaque environment where it is a source of information asymmetry between the seller and the marker makers. We further analyze how the presence of such seller affects other market participants, including market makers and end-user investors. We quantify the economic significance of the analytical predictions by calibrating the model.

Appendices

A Notation

The following schema summarizes the notation used throughout the paper.

β	Intertemporal discount rate used by all agents in the market.
λ	The arrival rate of end-user buy or sell investors (added up their arrival rate is 2λ).
ν	The duration of the large-seller's presence is exponentially distributed with parameter ν (i.e., mean duration is $1/\nu$).
θ	Vector that includes all deep parameters: $\theta = (\beta, \nu, N, \lambda, \eta, \delta, \omega)$.
η	The flow cost of carrying inventory for a market maker is ηi^2 where i denotes his inventory.
σ	The standard deviation of the security's fundamental value m_t.
δ	End-user investor demand sensitivity to price, see (2).
p_t^a	Ask price at time t.
p_t^b	Bid price at time t.
d	The duration of the large-seller's presence in the market (i.e., his time constraint).
i	A single market maker's inventory.
N	The number of market makers.
ω	The reservation value of buyers or sellers (i.e., maximum private value) expressed relative to common value m_t.
m_t[19]	The security's common value at time t (i.e., fundamental value).
q_t^a	Quantity offered at ask price at time t.
q_t^b	Quantity offered at bid price at time t.

B Stealth trading results

Lemma 8 (Market maker value under stealth trading)**.** *Market makers benefit from the presence of a large seller for any given large-seller liquidation intensity (f) if*

- *either the risk-bearing capacity of the market is sufficiently high (i.e, condition* (48) *in Appendix* C *holds which is the case when inventory-holding cost (η) is low enough, or when there are sufficiently many market makers (N), or when the investor arrival rate (λ) is high enough),*

- *or the duration of the large-seller's presence is sufficiently short or, more precisely, ν is sufficiently high.*

[19]"m" for martingale.

Lemma 8 echoes Lemma 1 for sunshine trading. The threshold risk-bearing capacity is the same as in sunshine trading. The duration threshold in the second bullet point could differ.

Lemma 9 (Expected path of market makers's inventory)**.** *The expected inventory of a market maker*

- *strictly increases and is concave in time while the large seller is executing,*
- *is homogeneous of degree one in the large seller's trade intensity,*
- *exhibits exponential decay towards zero after the seller disappears.*

The results in Lemma 9 echo the ones in Lemma 2. The mathematical expressions however differ and are available in the proofs of both lemmas.

C Proofs

Risk-capacity condition. The "risk-capacity condition" is formally defined as follows. Let $A^\star \in \left(0, \frac{\eta}{\beta}\right)$ be the unique positive solution to:

$$\eta - \beta A = \frac{8\delta\lambda A^2(1+\delta A)}{(N+1+2\delta A)^2}. \tag{47}$$

The risk-capacity condition holds if:

$$N^2 + \delta A^\star\,(2N-1) - \frac{1}{2}\,(1+\delta A^\star)\,(N+2\delta A^\star)^2\left(1 - \frac{(\beta A^\star)^2}{\eta^2}\right) \geq 0. \tag{48}$$

We refer to it as the risk-(bearing)-capacity condition because it holds when inventory-holding cost (η) is low enough (results in a small $A^\star > 0$), the number of market makers (N) is high enough, or investor arrivals (λ) are frequent (results in a small $A^\star > 0$).

Proof of Proposition 1. In the absence of the institutional investor, the quantities traded by each market maker are time homogenous and only depend on the size of that market maker's inventory:

$$q^a_{jt} = q^a(i_{jt}), \quad q^b_{jt} = q^b(i_{jt}),$$

where $q^a(i)$ and $q^b(i)$ are deterministic functions of the inventory. We can then rewrite the equation that governs the dynamics of i_{jt}, given in (8), as a time-homogenous autonomous equation:

$$\mathrm{d}i_{jt} = -q^a(i_{jt})\mathrm{d}N^B_t + q^b(i_{jt})\mathrm{d}N^S_t, \quad \forall t > 0.$$

Similarly, the trading revenue of each market maker follows the dynamics

$$\mathrm{d}W_{jt} = \left(\omega - \frac{1}{\delta}\sum_{n=1}^{N} q^a(i_{nt})\right) q^a(i_{jt})\mathrm{d}N_t^B - \left(\frac{1}{\delta}\sum_{n=1}^{N} q^b(i_{nt}) - \omega\right) q^b(i_{jt})\mathrm{d}N_t^S, \quad \forall t > 0.$$

By virtue of the dynamic programming principle, the value function $v_j^M(i)$ of the control problem solved by the j-th market maker is the solution to the Bellman equation:

$$\begin{aligned}\eta i^2 + \beta v_j^M(i) = \lambda \sup_{q_j^a, q_j^b} \Bigg[& \left(\omega - \frac{1}{\delta}\sum_{n=1}^{N} q_n^a\right) q_j^a + v_j^M(i - q_j^a) - \bar{v}_l^M(i) \\ & - \left(\frac{1}{\delta}\sum_{n=1}^{N} q_n^b - \omega\right) q_j^b + v_j^M(i + q_j^b) - v_j^M(i)\Bigg]\Bigg|_{q_n^a = q_j^{a,\star}, q_n^b = q_j^{b,\star} \text{ for all } n \neq j},\end{aligned} \tag{49}$$

where $(q_j^{a,\star}, q_j^{b,\star})$ is the optimizer of the Hamiltonian above, in which we set $q_n^a = q_j^{a,\star}, q_n^b = q_j^{b,\star}$ for all $n \neq j$ because we are considering a symmetric Markov perfect equilibrium. We make the ansatz that the value function $v_j^M(i)$ is quadratic and concave in i. Moreover, since all market makers are identical, value function and strategies are the same for all market makers, i.e.,

$$v_j^M(i) = v^M(i) = -Ai^2 + Bi + C, \tag{50}$$

for some constant $A > 0$. It then follows that the optimal control strategy for each market maker is given by

$$q^a = \frac{\omega + 2Ai - B}{\frac{N+1}{\delta} + 2A}, \qquad q^b = \frac{\omega - 2Ai + B}{\frac{N+1}{\delta} + 2A} \tag{51}$$

Using the expressions above, we can then rewrite (49) as

$$\frac{1}{\lambda}[(\eta - \beta A)i^2 + \beta Bi + \beta C] = \frac{8A^2 c(1 + \delta A)}{(N + 1 + 2\delta A)^2} i^2 + \frac{8\delta A(1 + \delta A)}{(N + 1 + 2\delta A)^2} Bi + \frac{2c(1 + \delta A)(B^2 + \omega^2)}{(N + 1 + 2\delta A)^2}.$$

By matching the coefficients of i^2, i and the constant term on both sides, we obtain that (A, B, C) must satisfy the following equations

$$\frac{1}{\lambda}(\eta - \beta A) = \frac{8\delta A^2(1+\delta A)}{(N+1+2\delta A)^2}, \tag{52}$$

$$\frac{\beta}{\lambda}B = \frac{8\delta A(1+\delta A)}{(N+1+2\delta A)^2}B, \tag{53}$$

$$\frac{\beta}{\lambda}C = \frac{2\delta(1+\delta A)(B^2+\omega^2)}{(N+1+2\delta A)^2}. \tag{54}$$

Hence, $A^\star$ must be the unique positive solution to (52), and $B^\star = 0$ otherwise (53) would not hold. Define

$$C^\star = \frac{\lambda}{\beta}\frac{2\delta(1+\delta A^\star)\omega^2}{(N+1+2\delta A^\star)^2} = \frac{\frac{\eta}{\beta A^\star} - 1}{4A^\star}\omega^2.$$

Then we obtain that a solution to (52)-(54) is given by $(A^\star, 0, C^\star)$. Hence, the value function is given by

$$\bar{v}(i) = -A^\star i^2 + C^\star.$$

The ask and bid price policy functions follow from the defining expression (7), and the expressions for the traded quantities given by (51) in which A and C are replaced, respectively, by the coefficients $A^\star$ and $C^\star$ of the optimal value function $v^M(i)$. Overall, for parameter $\theta = (\beta, \nu, N, \lambda, \eta, \delta, \omega)$, the results in Proposition 1 holds with positive constants:

$$A_\theta = \frac{\omega(1+2\delta A^\star)}{N+1+2\delta A^\star}, \quad B_\theta = \frac{2A^\star}{N+1+2\delta A^\star}, \quad C_\theta = \frac{\delta\omega}{N+1+2\delta A^\star},$$
$$D_\theta = \frac{2\delta A^\star}{N+1+2\delta A^\star}, \quad E_\theta = C^\star, \quad F_\theta = A^\star.$$

□

Proof of Corollary 1. We notice that the right hand side of equation (52) is strictly decreasing in N and strictly increasing in A, whereas the left hand side is strictly decreasing in A. Hence, the unique positive solution to (52), $A^\star$, is strictly increasing in N, and

$$\lim_{N\to\infty} A^\star = \frac{\eta}{\beta}. \tag{55}$$

It follows that

$$\lim_{N\to\infty} A_\theta = \lim_{N\to\infty} B_\theta = \lim_{N\to\infty} C_\theta = \lim_{N\to\infty} D_\theta = \lim_{N\to\infty} E_\theta = 0, \quad \lim_{N\to\infty} F_\theta = \frac{\eta}{\beta}.$$

Moreover, we also have

$$\lim_{N\to\infty} NC_\theta = \delta\omega, \quad \lim_{N\to\infty} ND_\theta = 2\delta\frac{\eta}{\beta}.$$

This completes the proof. □

Proof of Proposition 2. Given a constant liquidation intensity $f > 0$ and duration $d > 0$, the market is completely the same as in case of no large seller if $t > d$. Hence, the results are those given in Proposition 1. For $t \leq d$, by the dynamic programming principle the value function $v^{M☼}(i, f, t)$ of each market makers' control problem is the solution to the Bellman equation:

$$\begin{aligned}0 = -\eta i^2 - \beta v^{M,☼} + \frac{\partial v_t^{M,☼}}{\partial t} + \frac{\partial v_t^{M,☼}}{\partial i}\frac{f}{N} + \sup_{q_j^a, q_j^b}\Bigg[\lambda\left(\omega - \frac{1}{\delta}\sum_{n=1}^N q_n^a\right)q_j^a + \lambda v^{M,☼}(i - q_j^a, t) - \lambda v^{M,☼}(i,t) \\ -\left(-\omega + \frac{1}{\delta}\sum_{n=1}^N q_n^b\right)\left(\lambda q_j^b + \frac{f}{N}\right) + \lambda v_t(i + q_j^b, f) - \lambda v^{M,☼}(i,t)\Bigg]\Bigg|_{q_n^a = q_j^{a,\star}, q_n^b = q_j^{b,\star} \text{ for all } n\neq j},\end{aligned} \tag{56}$$

where $(q_j^{a,\star}, q_j^{b,\star})$ is the optimizer of Hamiltonian above in which we set $q_n^a = q_j^{a,\star}, q_n^b = q_j^{b,\star}$ for all $n \neq j$ because we are focusing symmetric Markov perfect equilibria. We make the following quadratic ansatz for the value function

$$v^{M,☼}(i, f, t) = -A(t)i^2 + B(t)i + C(t), \quad \forall t \geq 0, \tag{57}$$

where for notational simplicity we have suppressed f in the arguments of the right hand side above, and noted that the value function is the same for all market makers given that they have the same inventory costs and equally split the incoming trading flow. It then follows from the HJB equation (56) that the optimal control $(q_j^{a,\star}, q_j^{b,\star})$ is the same for each market marker, and given by

$$q^{a☼}(i) = \frac{\omega + 2A(t)i - B(t)}{\frac{N+1}{\delta} + 2A(t)}, \qquad q^{b☼}(i) = \frac{\omega - \frac{f}{\delta\lambda N} - 2A(t)i + B(t)}{\frac{N+1}{\delta} + 2A(t)},$$

as long as $A(t) > 0$. By using (57), and matching the coefficient of i^2 in Eq. (56), we obtain an ordinary differential equation

$$\frac{1}{\lambda}(\eta - \beta A(t)) + \frac{1}{\lambda}A'(t) = \frac{8c(A(t))^2(1 + \delta A(t))}{(N + 1 + 2\delta A(t))^2}, \quad A(d) = A^\star. \tag{58}$$

It follows from the definition of $A^\star$ given in Eq. (47) that $A'(d) = 0$. By the uniqueness of the solution to an ODE, we obtain that $A(t) \equiv A^\star$. Similarly, by matching the coefficient of i in

Eq. (56), we obtain an ODE for $B(t)$:

$$0 = -\frac{\eta}{A^\star} B(t) + B'(t) - f \frac{1}{\lambda} \frac{(\eta - \beta A^\star)}{2\delta A^\star} \frac{(N + 2\delta A^\star)}{N},$$

subject to the terminal condition $B(d) = 0$. The solution of the above ODE is

$$B(t) = -f \frac{(\frac{\eta}{A^\star} - \beta)}{2\delta\lambda} \frac{(N + 2\delta A^\star)}{N} \frac{1 - e^{-\frac{\eta}{A^\star}(d-t)}}{\eta/A^\star}, \quad \forall t \in [0, d]. \tag{59}$$

The function $C(t)$ can be determined similarly: for $t \in [0, d]$,

$$C(t) = E_\theta + 1_{\{t \le d\}} \Bigg[f \frac{2(1 + \delta A^\star)(N + 2\delta A^\star)}{N(N + 1 + 2\delta A^\star)^2} \int_0^{d-t} e^{-\beta(d-t-s)} (f B(s) + \omega)\, \mathrm{d}s$$
$$+ f^2 \left(\frac{N^2 + \delta A^\star(2N - 1)}{\delta\lambda N^2 (N + 1 + 2\delta A^\star)^2} \frac{1 - e^{-\beta(d-t)}}{\beta} + \frac{2\delta\lambda(1 + \delta A^\star)}{(N + 1 + 2\delta A^\star)^2} \int_0^{d-t} t e^{-\beta(d-t-s)} B^2(s) \mathrm{d}s \right) \Bigg],$$

where E_θ is given in the proof of Proposition 1.

Finally, using the expressions for the ask and bid price policy functions in the absence of liquidation, the expression for the optimal trading quantities, and the expressions for the value function given in the proof of Proposition 1, we obtain that the ask and bid price policy functions in the presence of liquidation are given by

$$p^{a☼}(i, f, t) = p^a(i) - 1_{\{t \le d\}} f \left(G_\theta^☼(d - t) \right),$$
$$p^{b☼}(i, f, t) = p^b(i) - 1_{\{t \le d\}} f \left(G_\theta^☼(d - t) + H_\theta^☼ \right),$$

where $G_\theta^☼(t) = -N B(t)/(N + 1 + 2\delta A^\star)$ and $H_\theta^☼ = 1/(\delta\lambda(N + 1 + 2\delta A^\star))$. The optimal control is given by

$$q^{a☼}(i, f, t) = q^a(i) + 1_{t \le d} f \frac{\delta}{N} \left(G_\theta^☼(d - t) \right),$$
$$q^{b☼}(i, f, t) = q^b(i) - 1_{t \le d} f \frac{\delta}{N} \left(G_\theta^☼(d - t) + H_\theta^☼ \right),$$

Finally, the value function takes the form

$$v^{M☼}(i, f, t) = v^M(i) + 1_{\{t \le d\}} \left(I_\theta^☼ (d - t, f) + f J_\theta^☼ (d - t)\, i \right),$$

where $I_\theta^☼(d - t, f) = C(t) - E_\theta$, which is strictly increasing in variable "$d - t$" for $t \le d$ and strictly convex in f, and $J_\theta^☼(d - t) = B(t)$, which is negative and increasing in t. This completes the proof. □

Proof of Lemma 1. We first show that $I_\theta^{☼}(d, f) > 0$ for $d > 0$ sufficiently small. It suffices to show that $I_1(d, f) := e^{\beta d} I_\theta^{☼}(d, f) > 0$ under the condition (48). Because $I_1(0, f) = I_\theta^{☼}(0, f) = 0$, it is enough to show that $\frac{\partial}{\partial d} I_1(d, f) > 0$, or equivalently, $e^{-\beta d}\frac{\partial}{\partial d} I_1(d, f) > 0$ for d close to 0. To that end, by Taylor expansion (up to the 0th order), we obtain that

$$e^{-\beta d}\frac{\partial}{\partial d} I_1(d, f) \approx f\frac{2(1+\delta A^\star)(N+2\delta A^\star)}{N(N+1+2\delta A^\star)^2}\omega + f^2\frac{N^2+\delta A^\star(2N-1)}{\delta\lambda N^2(N+1+2\delta A^\star)^2}, \tag{60}$$

if $d \approx 0$. Since the expression in the last line of (60) is positive for any $f > 0$, the claim for small d is proven.

Next, we prove the other claim. It is easily seen from the first two lines of Eq. (60) that the coefficient of f^2 is strictly decreasing in d, so the coefficient is bounded below by

$$\frac{-\frac{1}{2}(1+\delta A^\star)(N+2\delta A^\star)^2\left(1-\frac{(\beta A^\star)^2}{\eta^2}\right)+N^2+\delta A^\star(2N-1)}{\delta\lambda N^2(N+1+2\delta A^\star)^2}.$$

Hence, if (48) holds, for any $f > 0$ we have $\frac{\partial}{\partial d} I_1(d, f) > 0$ for all $d > 0$. Hence, $I_\theta^{☼}(d, f) = e^{-\beta d} I_1(d, f) > 0$. This completes the proof. □

Proof of Lemma 2. We first derive an explicit expression for the expected inventory. We use (8), (27) and (28) to obtain the dynamics of a market maker's inventory:

$$\begin{aligned}\mathrm{d}i_{jt} =& 1_{\{t\le d\}}\frac{f}{N}\mathrm{d}t - \left(C_\theta + D_\theta i_{jt} + 1_{\{t\le d\}}\frac{f}{N}G_\theta^{☼}(d-t)\right)\delta \mathrm{d}N_t^B \\ &+ \left(C_\theta - d_\theta i_{jt} - 1_{\{t\le d\}}\frac{f}{N}\left(G_\theta^{☼}(d-t)+H_\theta^{☼}\right)\right)\delta \mathrm{d}N_t^S.\end{aligned}$$

Using the compensation formula, we deduce from the above that the function $\mathbb{E}[i_{jt}]$ satisfies the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[i_{jt}] = -2\delta\lambda D_\theta \mathbb{E}[i_{jt}] + 1_{\{t\le d\}}\frac{f}{N}\left(1-\delta\lambda H_\theta^{☼} - 2\delta\lambda G_\theta^{☼}(d-t)\right),$$

subject to the initial condition $i_{j0} = 0$. Solving the ODE we obtain that $\mathbb{E}[i_{jt}] = f \times g(t)$, where

$$g(t) = \begin{cases} \frac{N+2\delta A^\star}{N}\frac{M}{4\delta\lambda A^\star}\left[\frac{\beta A^\star}{\eta}\frac{1-e^{-Mt}}{M} + \left(1-\frac{\beta A^\star}{\eta}\right)\frac{e^{\frac{\eta}{A^\star}t}-e^{-Mt}}{\frac{\eta}{A^\star}+M}\right], & t \le d, \\ g(d)e^{-M(t-d)}, & t > d,\end{cases}$$

where $M = \frac{4\delta\lambda A^\star}{N+1+2\delta A^\star}$. A straightforward calculation yields $g'(t) > 0$ for $t \le d$. This completes the proof. □

Proof of Lemma 3. By combining (25) with the explicit expression for $\mathbb{E}[i_{jt}]$ in the proof of Lemma 2, we obtain that the expected bid price pressure under sunshine trading is given as follows: for $t \leq d$,

$$\mathbb{E}\left[p^{b☼}(i_{jt}, f, t)\right] = -\frac{\omega(1+2\delta A^\star)}{N+1+2\delta A^\star} - \frac{f}{2\delta\lambda(N+1+2\delta A^\star)} \times \left[2 + (N+2\delta A^\star)\left(1 - \frac{\frac{\eta}{A^\star} - \beta}{M + \frac{\eta}{A^\star}} e^{-\frac{\eta}{A^\star}(d-t)} - \frac{\beta A^\star}{\eta} e^{-Mt} - \frac{\frac{\eta}{A^\star} - \beta}{\frac{\eta}{A^\star}} \frac{M}{M+\frac{\eta}{A^\star}} e^{-\frac{\eta}{A^\star}d - Mt}\right)\right], \quad (61)$$

for $t > d$,

$$\mathbb{E}\left[p^{b☼}(i_{jt}, f, t)\right] = -\frac{\omega(1+2\delta A^\star)}{N+1+2\delta A^\star} - \frac{f(N+2\delta A^\star)}{2\delta\lambda(N+1+2\delta A^\star)} \times \left(\frac{\beta A^\star}{\eta}\left[1 - e^{-Md}\right] + \frac{(\eta - \beta A^\star)}{\eta} \frac{M}{M+\frac{\eta}{A^\star}}\left[1 - e^{-(M+\frac{\eta}{A^\star})d}\right]\right) e^{-M(t-d)},$$

where $M = \frac{4\delta\lambda A^\star}{N+1+2\delta A^\star}$. Straightforward calculation yields that $\mathbb{E}[p^{b☼}(i_{jt}, t)]$ is strictly convex in t for $t < d$ (See the gray solid line in the bottom panel of Figure 3). Moreover, because[20]

$$\frac{\partial}{\partial t}\mathbb{E}\left[p^{b☼}(i_{jt}, f, t)\right]\Big|_{t=0} \propto \frac{\eta - \beta A^\star}{MA^\star + \eta} \frac{\eta}{A^\star} e^{-\frac{\eta}{A^\star}d} - \left(\frac{\beta A^\star}{\eta} + \frac{\eta - \beta A^\star}{\eta} \frac{M}{M+\frac{\eta}{A^\star}} e^{-\frac{\eta}{A^\star}d}\right) M.$$

Clearly, the right hand side of the above expression is monotone in d, and is negative if $d > 0$ is sufficiently large. As $d \to 0$, this expression converges to

$$\frac{-2\beta + \frac{\eta}{A^\star} - M}{M + \frac{\eta}{A^\star}} = \frac{1}{M+\frac{\eta}{A^\star}}\left(-\beta - M\frac{N-1}{N+1+2\delta A^\star}\right) < 0.$$

Thus, we know that for all $d > 0$ it holds that $\frac{\partial}{\partial t}\mathbb{E}[p^{b☼}(i_{jt}, t)]|_{t=0} < 0$, and the expected bid price pressure is decreasing in t for small t, as claimed. Moreover,

$$\frac{\partial}{\partial t}\mathbb{E}\left[p^{b☼}(i_{jt}, f, t)\right]\Big|_{t=d} \propto \frac{\eta - \beta A^\star}{MA^\star + \eta} \frac{\eta}{A^\star} - \left(\frac{\beta A^\star}{\eta} + \frac{\eta - \beta A^\star}{\eta} \frac{M}{M+\frac{\eta}{A^\star}} e^{-\frac{\eta}{A^\star}d}\right) M e^{-Md}.$$

Thus, if $d > 0$ is sufficiently large, we have $\frac{\partial}{\partial t}\mathbb{E}[p^{b☼}(i_{jt}, t)]|_{t=d} > 0$, so the expected bid price pressure is increasing in t if t and d are both sufficiently large. The properties for bid price pressure after d hold obviously. For ask price, we obtain the results via relationship

$$\mathbb{E}\left[p^{a☼}(i_{jt}, f, t)\right] = \mathbb{E}\left[p^{b☼}(i_{jt}, f, t)\right] + 2A_\theta + f H_\theta^{☼} 1_{\{t \leq d\}}.$$

[20] The symbol $\propto$ means proportional to, i.e., the left hand side is a positive constant multiple of the right hand side.

This completes the proof. □

Proof of Proposition 3. Prior to the end of liquidation the market maker faces, at any point in time, the same average amount of time left. This is due to the memoryless property of the exponential distribution of d. Specifically, the market maker treats the duration of the liquidation as an independent exponential random variable with mean $1/\nu$. Hence, before d, the value function and the strategy of each market maker, as well as the price policies, can be obtained by taking the expectation of the corresponding quantities in Proposition 2 and assuming the distribution $d - t \sim \text{Exp}(\nu)$ in the event $\{t \leq d\}$. In particular, one can show that

$$G_\theta^{\text{☁}} = -\frac{\eta - \beta A^\star}{2\delta\lambda A^\star} \frac{N + 2\delta A^\star}{N + 1 + 2\delta A^\star} \frac{A^\star}{\nu A^\star + \eta}, \tag{62}$$

$$I_\theta^{\text{☁}}(f) = \int_0^\infty \nu e^{-\nu u} I_\theta^{\text{☼}}(u, f) \mathrm{d}u, \tag{63}$$

$$J_\theta^{\text{☁}} = -\frac{\eta - \beta A^\star}{2\delta\lambda A^\star} \frac{N + 2\delta A^\star}{N} \frac{A^\star}{\nu A^\star + \eta}. \tag{64}$$

After time d, there is no liquidation. Hence, the value function and the strategy of any market maker, as well as the corresponding price policies, are those given in Proposition 1. □

Proposition of Lemma 4. By combining (31) with Lemma 9, we obtain that, for $t \leq d$,

$$\mathbb{E}\left[p^{b\text{☁}}(i_{jt}, f, t)\right] = \frac{-\omega(1 + 2\delta A^\star)}{N + 1 + 2\delta A^\star} - \frac{f\left[2 + (N + 2\delta A^\star)\left(\frac{\eta - \beta A^\star}{\eta + \nu A^\star} + \frac{1}{2A^\star}\frac{\beta A^\star + \nu A^\star}{\eta + \nu A^\star}(1 - e^{-Mt})\right)\right]}{2\delta\lambda(N + 1 + 2\delta A^\star)}, \tag{65}$$

for $t > d$,

$$\mathbb{E}\left[p^{b\text{☁}}(i_{jt}, f, t)\right] = \frac{-\omega(1 + 2\delta A^\star)}{N + 1 + 2\delta A^\star} - \frac{f(N + 2\delta A^\star)\left(\frac{1}{2A^\star}\frac{\beta A^\star + \nu A^\star}{\eta + \nu A^\star}(1 - e^{-Md})\right)}{2\delta\lambda(N + 1 + 2\delta A^\star)} e^{-M(t-d)},$$

where $M = \frac{4\delta\lambda A^\star}{N + 1 + 2\delta A^\star}$. For the ask price, we have

$$\mathbb{E}\left[p^{a\text{☁}}(i_{jt}, f, t)\right] = \mathbb{E}\left[p^{b\text{☁}}(i_{jt}, f, t)\right] + 2A_\theta + f\, H_\theta^{\text{☼}} 1_{\{t \leq d\}}.$$

The properties follow immediately. □

Proof of Lemma 5. Under sunshine trading, using (4) and (61), we can express the objective of the institutional investor in (36) as

$$\sup_{f\geq 0}\Big(P(d)\,f - Q(d)\,f^2\Big), \tag{66}$$

where $P(d)$ and $Q(d)$ are positive functions given by

$$P(d) = \frac{N\omega}{N+1+2\delta A^\star} h_\beta(d), \tag{67}$$

$$Q(d) = \frac{h_\beta(d)/(2\delta\lambda)}{N+1+2\delta A^\star}\Bigg\{(N+2+2\delta A^\star) + (N+2\delta A^\star)\Bigg[\frac{\beta}{M+\frac{\eta}{A^\star}} - \frac{\beta A^\star}{\eta}\frac{h_{M+\beta}(d)}{h_\beta(d)} - \frac{\beta}{(M+\beta)}\frac{h_{\frac{\eta}{A^\star}}(d)}{h_\beta(d)} - \frac{(\eta-\beta A^\star)}{\eta}\frac{M}{M+\frac{\eta}{A^\star}}\frac{M+\beta+\frac{\eta}{A^\star}}{M+\frac{\eta}{A^\star}}\frac{h_{M+\beta+\frac{\eta}{A^\star}}(d)}{h_\beta(d)}\Bigg]\Bigg\}, \tag{68}$$

with $M=\frac{4\delta\lambda A^\star}{N+1+2\delta A^\star}$ and $h_\gamma(u)=\frac{1-e^{-\gamma u}}{\gamma}$ for any positive parameter $\gamma>0$. It follows that the optimal sunshine liquidation intensity is given by

$$f^{☼}(d) = \frac{P(d)}{2Q(d)}.$$

To prove the monotonicity of $f^{☼}(d)$ in d, we notice that

$$\begin{aligned}\frac{1}{f^{☼}(d)} &= \frac{2Q(d)}{P(d)}\\ &= \frac{1}{\delta\lambda N\omega}\Bigg\{(N+2+2\delta A^\star) + (N+2\delta A^\star)\Bigg[\frac{(\frac{\eta}{A^\star}-\beta)}{\frac{\eta}{A^\star}+M}\frac{\beta A^\star}{\eta-\beta A^\star} - \frac{\beta A^\star}{\eta}\frac{h_{M+\beta}(d)}{h_\beta(d)}\\ &\quad - \frac{\beta}{(M+\beta)}\frac{h_{\frac{\eta}{A^\star}}(d)}{h_\beta(d)} - \frac{(\eta-\beta A^\star)}{\eta}\frac{M}{M+\frac{\eta}{A^\star}}\frac{M+\beta+\frac{\eta}{A^\star}}{M+\beta}\frac{h_{M+\beta+\frac{\eta}{A^\star}}(d)}{h_\beta(d)}\Bigg]\Bigg\}\end{aligned}$$

Next, let us state (without proof) the following simple result:

Lemma 10. *For $\beta_1>\beta_2$, the function $h_{\beta_1}(d)/h_{\beta_2}(d)$ is strictly decreasing in d.*

Then we know that $\frac{1}{f^{☼}(d)}$ is strictly increasing. Hence, the optimal liquidation intensity $f^{☼}(d)$, which is known to be positive, is strictly decreasing in d. To prove the bounds in (38), we first calculate the limits

$$\lim_{d\to 0} f^{☼}(d) = \frac{\delta\lambda N\omega}{N+2\delta A^\star+2}\left\{1 - \frac{N+2\delta A^\star}{N+2\delta A^\star+2}k(0)\right\}^{-1},$$

$$\lim_{d\to\infty} f^{☼}(d) = \frac{\delta\lambda N\omega}{N+2\delta A^\star+2}\left\{1-\frac{N+2\delta A^\star}{N+2\delta A^\star+2}k(\infty)\right\}^{-1},$$

where $k(\nu) = \frac{(\nu+\beta)^2}{(\nu+\frac{\eta}{A^\star})(M+\nu+\beta)}$. Hence, the upper and lower bounds follow from the fact that $k(\nu)$ is strictly increasing in ν (a fact that can be easily checked by taking first-order derivatives).

Under stealth trading, using (4) and (65) we can express the objective of the institutional investor in (37) as

$$\sup_{f\geq 0}\left(\widetilde{P} f - \widetilde{Q} f^2\right), \tag{69}$$

where $\widetilde{P}, \widetilde{Q}$ are positive constants given by

$$\widetilde{P} = \frac{1}{\nu+\beta}\frac{N\omega}{N+1+2\delta A^\star}, \tag{70}$$

$$\widetilde{Q} = \frac{1}{N+1+2\delta A^\star}\frac{1}{2\delta\lambda}\left((N+2\delta A^\star+2)\frac{1}{\nu+\beta} - (N+2\delta A^\star)\frac{\nu+\beta}{\nu+\frac{\eta}{A^\star}}\frac{1}{M+\nu+\beta}\right). \tag{71}$$

Hence, the optimal stealth liquidation intensity is given by

$$f^{\circ} = \frac{\widetilde{P}}{2\widetilde{Q}} = \frac{\delta\lambda N\omega}{N+2\delta A^\star+2}\left\{1-\frac{N+2\delta A^\star}{N+2\delta A^\star+2}k(\nu)\right\}^{-1}.$$

The inequality in the middle of (38) again follows from the monotonicity of $k(\nu)$. □

Proof of Proposition 4. To prove (39), we will make use of the following result:

Lemma 11. *It holds that*

$$\int_0^\infty \nu e^{-\nu s} P(s)\mathrm{d}s = \widetilde{P}, \tag{72}$$

$$\int_0^\infty \nu e^{-\nu s} Q(s)\mathrm{d}s < \widetilde{Q}, \tag{73}$$

where $\widetilde{P}$ and $\widetilde{Q}$ are positive constants defined in (70) and (71), $P(s)$ and $Q(s)$ are positive functions of s defined in (67) and (68).

Proof. It is easily seen that the equality in (72) holds. To prove (73), it is straightforward to verify that $\int_0^\infty \nu e^{-\nu s}Q(s)\mathrm{d}s - \widetilde{Q}$ is a positive constant multiple of $-(\eta-\beta A^\star) < 0$. □

We notice that the first inequality of (39) holds obviously. To prove the second inequality, we have

$$
\begin{aligned}
\mathbb{E}_d\left[P_d\left(f^{\text{☁}}\right)|d \text{ revealed}\right] &= \int_0^\infty \nu e^{-\nu s}\left(P(s)\, f^{\text{☁}} - Q(s)\left(f^{\text{☁}}\right)^2\right) \mathrm{d}s \\
&> \widetilde{P} f^{\text{☁}} - \widetilde{Q}(f^{\text{☁}})^2 = \mathbb{E}_d\left[P_d\left(f^{\text{☁}}\right)|d \text{ hidden}\right],
\end{aligned}
$$

where we have used Lemma 11 to deduce the inequality above. □

Proof of Lemma 6 First of all, (40) has already been proven in Corollary 1. To prove (41), we make use of the following result:

Lemma 12.

$$
\lim_{N\to\infty} \frac{f^{\text{☁}}}{N} = \frac{\omega}{4\frac{\eta}{\beta(\nu+\beta)} + \frac{2}{\delta\lambda}}
$$

Proof. As $N \to \infty$, it follows that $A^\star \uparrow \frac{\eta}{\beta}$ (see (55)). Moreover, from the equation that defines $A^\star$, we have that

$$
\frac{\eta}{A^\star} - \beta = \frac{8\delta\lambda A^\star(1+\delta A^\star)}{(N+1+2\delta A^\star)^2},
$$

so as $N \to \infty$, $\frac{\eta}{A^\star} \to \beta$ and $(N+1+2\delta A^\star)(\frac{\eta}{A^\star} - \beta) \to 0$. Hence,

$$
\lim_{N\to\infty}\left(N + 2\delta A^\star + 2 - (N+2\delta A^\star)\frac{\nu+\beta}{\nu+\frac{\eta}{A^\star}}\frac{\nu+\beta}{\nu+\beta+M}\right) = \delta\lambda\left(\frac{2}{\delta\lambda} + \frac{4\eta}{\beta(\nu+\beta)}\right).
$$

It follows that

$$
\lim_{N\to\infty}\frac{f^{\text{☁}}}{N} = \frac{\delta\lambda\omega}{\lim_{N\to\infty}(N+2\delta A^\star + 2 - (N+2\delta A^\star)\frac{\nu+\beta}{\nu+\frac{\eta}{A^\star}}\frac{\nu+\beta}{\nu+\beta+M})} = \frac{\omega}{4\frac{\eta}{\beta(\nu+\beta)} + \frac{2}{\delta\lambda}}.
$$

□

Using Proposition 3, (63), (64) and Lemma 12, one can easily show that

$$
\begin{aligned}
\lim_{N\to\infty}\frac{f^{\text{☁}}}{N}\cdot NJ_\theta^{\text{☁}} &= 0, \\
\lim_{N\to\infty} I_\theta^{\text{☁}}(f^{\text{☁}}) &= \frac{1}{\delta\lambda(\nu+\beta)}\left(\frac{\omega}{4\frac{\eta}{\beta(\nu+\beta)}+\frac{2}{\delta\lambda}}\right)^2.
\end{aligned}
$$

This completes the proof for (41). To prove (42), we make use of the following result:

Lemma 13. *For any fixed $d > 0$,*

$$\lim_{N\to\infty} \frac{f^{☼}(d)}{N} = \frac{\omega}{\frac{2}{\delta\lambda} + \frac{4\eta}{\beta^2}(1 - \frac{\beta d}{e^{\beta d}-1})}.$$

Proof. To prove the result, let us define

$$g_N(d) = \frac{\beta}{M + \frac{\eta}{A^\star}} - \frac{\beta A^\star}{\eta}\frac{h_{M+\beta}(d)}{h_\beta(d)} - \frac{\beta}{(M+\beta)}\frac{h_{\frac{\eta}{A^\star}}(d)}{h_\beta(d)} - \frac{(\eta - \beta A^\star)}{\eta}\frac{M}{M + \frac{\eta}{A^\star}}\frac{M + \beta + \frac{\eta}{A^\star}}{M + \beta}\frac{h_{M+\beta+\frac{\eta}{A^\star}}(d)}{h_\beta(d)},$$

where $M = \frac{4\delta\lambda A^\star}{N+1+2\delta A^\star}$ and $h_\gamma(u) = \frac{1-e^{-\gamma u}}{\gamma}$ for any positive parameter $\gamma > 0$. Then we can write $\frac{f^{☼}(d)}{N}$ as

$$\frac{f^{☼}(d)}{N} = \frac{\delta\lambda\omega}{N + 2\delta A^\star + 2 + (N + 2\delta A^\star)g_N(d)}.$$

For any fixed $d > 0$, by the fact that $M \to 0$ and $\frac{\eta}{A^\star} \to \beta$ as $N \to \infty$, and that $h_\gamma(d)$ is continuous in the parameter $q > 0$, we deduce that $\lim_{N\to\infty} g_N(d) = -1$. Thus, we can write the denominator of $\frac{f^{☼}(d)}{N}$ as

$$1 - g_N(d) + (N + 2\delta A^\star + 1)(1 + g_N(d)). \tag{74}$$

The first two terms of (74) clearly converge to 2 as $N \to \infty$. The last term involves a "$\infty \cdot 0$" type limit, and hence requires more care. To that end, we use the equation that defines $A^\star$ to express N as a "function" of $A^\star$:

$$N + 1 + 2\delta A^\star = A^\star \sqrt{\frac{8\delta\lambda(1 + \delta A^\star)}{\theta - \beta A^\star}}.$$

Then we can write

$$M = \frac{4\delta\lambda A^\star}{N + 1 + 2\delta A^\star} = \sqrt{\frac{2\delta\lambda(\theta - \beta A^\star)}{1 + \delta A^\star}}.$$

Using the above result, the limit of $(N + 2\delta A^\star + 1)(1 + g_N(d))$ as $N \to \infty$ may be written as the limit of a function of $A^\star$, as $A^\star \to \frac{\eta}{\beta}$. With help of the Mathematica software, we can evaluate this limit as

$$\lim_{N\to\infty}[1 - g_N(d) + (N + 2\delta A^\star + 1)(1 + g_N(d))] = \delta\lambda\left[\frac{2}{\delta\lambda} + \frac{4\eta}{\beta^2}\left(1 - \frac{\beta d}{e^{\beta d} - 1}\right)\right].$$

It follows that

$$\lim_{N\to\infty}\frac{f^{☼}(d)}{N} = \frac{\delta\lambda\omega}{\lim_{N\to\infty}[1 - g_N(d) + (N + 2\delta A^\star + 1)(1 + g_N(d))]} = \frac{\omega}{\frac{2}{\delta\lambda} + \frac{4\eta}{\beta^2}(1 - \frac{\beta d}{e^{\beta d}-1})}.$$

□

Using Proposition 2, Lemma 13 and the explicit expressions for $J_\theta^{☼}(d)$ and $I_\theta^{☼}(d, f)$ that appeared in the proof of Proposition 2, one can easily show that, for $t \leq d$,

$$\lim_{N\to\infty} \frac{f^{☼}(d)}{N} \cdot N J_\theta^{☼}(d-t) = 0,$$

$$\lim_{N\to\infty} I_\theta^{☼}(d-t, f^{☼}(d)) = \frac{1}{\delta\lambda}\left(\frac{\omega}{4\frac{\eta}{\beta^2}(1-\frac{\beta d}{e^{\beta d}-1}) + \frac{2}{\delta\lambda}}\right)^2 \frac{1-e^{-\beta(d-t)}}{\beta}.$$

This completes the proof for (42). □

Proof of Corollary 2 From Lemma 12 and Lemma 13 we know that the optimal liquidation intensity will diverge to ∞ roughly linearly in N, the number of market makers. When the liquidation intensity is exogenously fixed as $f^e(d)$, then using the limiting result

$$\lim_{N\to\infty} \frac{f^e(d)}{N} = 0,$$

we can easily obtain (43) via similar arguments to those used in Lemma 6. □

Proof of Lemma 7. Let $Volume_t$ be the total trading volume of one market maker from time 0 to time t, then for liquidation intensity $f = f^{☼}(d)$, we have $Volume_0 = 0$ and for $t \leq d$,

$$\begin{aligned} \mathrm{d}\, Volume_t =& \frac{f}{N}\mathrm{d}t + q^{a☼}(i_{jt}, f, t)\mathrm{d}N_t^B + q^{b☼}(i_{jt}, f, t)\mathrm{d}N_t^S \\ =& \left[\left(\frac{N+2\delta A^\star}{N+1+2\delta A^\star}\right)\frac{f}{N} + \frac{2\omega\delta\lambda}{N+1+2\delta A^\star}\right]\mathrm{d}t + \mathrm{d}\text{``martingale''}. \end{aligned}$$

It follows that

$$\mathbb{E}[Volume_d] = d \times \left[\left(\frac{N+2\delta A^\star}{N+1+2\delta A^\star}\right)\frac{f}{N} + \frac{2\omega\delta\lambda}{N+1+2\delta A^\star}\right].$$

The result immediately follows. □

Proof of Proposition 5. Let us denote the expected surplus without liquidation by

$$v^I(i,t) = \mathbb{E}\left[\int_t^\infty e^{-\beta(s-t)}\left(\frac{1}{2}\delta\,(\omega - p_s^a)^2\,\mathrm{d}N_s^B + \frac{1}{2}\delta\left(\omega + p_s^b\right)^2\mathrm{d}N_s^S\right)|i_{jt} = i\right].$$

When there is no liquidation, the market makers use stationary strategies, so $v^I(t,i)$ will be independent of time t. In this case, $v^I(0,i)$ solves the equation

$$\begin{aligned}&\beta v^I(i,0)\\ &=\lambda\Big(\frac{\delta}{2}(\omega-p_t^a)^2+v^I(i-\frac{\delta}{N}(\omega-p_t^a),0)-v^I(i,0)+\frac{\delta}{2}(\omega+p_t^b)^2+v^I(i+\frac{\delta}{N}(p_t^b+\omega),0)-v^I(i,0)\Big),\end{aligned}$$

where $p_t^a = p^a(i)$ and $p_t^b = p^b(i)$ are given in (11) and (12), respectively. In this case, following a similar argument to that in the proof of Proposition 1, we obtain that

$$v^I(i) = E_\theta^I + F_\theta^I i^2,$$

where

$$E_\theta^I = \frac{\delta\lambda(N^2+2\delta A_\omega)}{\beta(N+1+2\delta A^\star)^2}\omega^2, \quad F_\theta^I = A_\omega,$$

with

$$A_\omega = \frac{4\delta N^2(A^\star)^2\lambda}{(\beta+2\lambda)(N+1+2\delta A^\star)^2-2\lambda(N+1)^2} > 0, \tag{75}$$

On the other hand, a direct calculation of the expectations appeared the welfare function yields the same result. Hence, we have derived the end-investor's welfare when the large seller is not present. Notice that this also gives the welfare of the end-investor for $t > d$ if the liquidation duration is $d > 0$.

Similarly, when a sunshine liquidation with liquidation intensity $f > 0$ and duration $d > 0$ is in place, we denote by $v^{I☼}$ the surplus of end-user investors, then $v^{I☼}(i,t)$ solves

$$\begin{aligned}0=&-\beta v^{I☼}+\frac{\partial v^{I☼}}{\partial t}+\frac{\partial v^{I☼}}{\partial i}\frac{f}{N}+\lambda\Big(\frac{\delta}{2}(\omega-p_t^a)^2+v^{I☼}(i-\frac{\delta}{N}(\omega-p_t^a),t)-v^{I☼}(i,t)\\ &+\frac{\delta}{2}(\omega+p_t^b)^2+v^{I☼}(i+\frac{\delta}{N}(p_t^b+\omega),t)-v^{I☼}(i,t)\Big),\end{aligned}$$

for $t \in [0,d]$, where $p_t^a = p^{a☼}(i,t)$ and $p_t^b = p^{b☼}(i,t)$ are given in (24) and (25), respectively.

Following the argument used in the proof of Proposition 2, we can show that

$$v^{I☼}(i,t) = v^I(i) + 1_{\{t\le d\}}\left(I_\theta^{I☼}(d-t,f) + J_\theta^{I☼}(d-t)f\,i\right),$$

where

$$J_\theta^{I☼}(u) = \int_0^u e^{-(\beta+M)(u-s)}\cdot\frac{2}{N}\Big(A_\omega + \frac{[N^2A^\star-(N+1)A_\omega](1-2\delta\lambda NB(s))}{(N+1+2\delta A^\star)^2}\Big)\mathrm{d}s, \tag{76}$$

where $B(s)$ is defined in (59). Because $B(s) < 0$ for all $s > 0$, and

$$N^2A^\star - (N+1)A_\omega = N^2A^\star(N+1+2\delta A^\star)\frac{\beta(N+1+2\delta A^\star)+4\delta\lambda A^\star}{(\beta+2\lambda)(N+1+2\delta A^\star)^2 - 2\lambda(N+1)^2} > 0,$$

we know that

$$J_\theta^{I☼}(u) > \frac{2}{N}A_\omega \int_0^u e^{-(\beta+M)(u-s)}\mathrm{d}s > 0.$$

Moreover,

$$\begin{aligned} I_\theta^{I☼}(u,f) = &- f\frac{\omega(N^2+2\delta A_\omega)}{N(N+1+2\delta A^\star)^2}\frac{1-e^{-\beta u}}{\beta} + f^2\int_0^u e^{-\beta(u-s)}\frac{(1+\frac{2\delta A^\star}{N}+2\delta\lambda B(s))J_\theta^{I☼}(s)}{N+1+2\delta A^\star}\mathrm{d}s \\ &+ f^2\int_0^u e^{-\beta(u-s)}\frac{(N^2+2\delta A_\omega)(1-2\delta\lambda N B(s)+2\delta^2\lambda^2N^2B^2(s))}{2\delta\lambda N^2(N+1+2\delta A^\star)^2}\mathrm{d}s, \end{aligned} \tag{77}$$

Because for any $s > 0$ we know that $J_\theta^{I☼}(s) > 0$, and $0 > 2\delta\lambda B(s) > -\frac{\eta-\beta A^\star}{\eta}(1+\frac{2\delta A^\star}{N}) > -(1+\frac{2\delta A^\star}{N})$, we deduce that the coefficient of f^2 in (77) is positive, while that of f in (77) is negative. Hence, there exists $f > 0$ such that $I_\theta^{I☼}(d,f) > 0$ if and only if $f > f$, and $I_\theta^{I☼}(d,f) < 0$ if and only if $0 < f < f$. This completes the proof for the case of sunshine trading.

The case of stealth trading can be treated similarly. After standard calculations, one obtains that the end-user investor's surplus, denoted by $v^{I☁}(i,t)$, is given by

$$v^{I☁}(i,t) = v^I(i) + 1_{\{t\le d\}}\left(I_\theta^{I☁}(f) + J_\theta^{I☁} f\, i\right),$$

where

$$J_\theta^{I☁} = \frac{\nu}{\nu+\beta+M}\int_0^\infty e^{-\nu s}\cdot\frac{2}{N}\Big(A_\omega + \frac{(N^2A^\star-(N+1)A_\omega)(1-2\delta\lambda N B(s))}{(N+1+2\delta A^\star)^2}\Big)\mathrm{d}s. \tag{78}$$

The latter quantity can be shown (as above for the case of sunshine trading) to be a positive constant, and

$$\begin{aligned} I_\theta^{I☁}(f) = &- f\frac{\omega(N^2+2\delta A_\omega)}{N(N+1+2\delta A^\star)^2}\frac{\nu}{\nu+\beta} + f^2\frac{\nu}{\nu+\beta}\int_0^\infty e^{-\nu s}\frac{(1+\frac{2\delta A^\star}{N}+2\delta\lambda B(s))J_\theta^{I☼}(s)}{N+1+2\delta A^\star}\mathrm{d}s \\ &+ f^2\frac{\nu}{\nu+\beta}\int_0^\infty e^{-\nu s}\frac{(N^2+2\delta A_\omega)(1-2\delta\lambda N B(s)+2\delta^2\lambda^2N^2B^2(s))}{2\delta\lambda N^2(N+1+2\delta A^\star)^2}\mathrm{d}s, \end{aligned} \tag{79}$$

which is again a convex, quadratic function of f with a negative linear coefficient. Hence, we obtain qualitatively the same result as in the sunshine trading. □

Proof of Proposition 6. Consider a social planner that maximizes the aggregate utility of the institutional investor, end-investors and the market makers. The social planner controls the price policy functions, but the liquidation intensity is still decided by the large seller. Then $\frac{1}{N}$-th of the total utility W_t follows

$$
\begin{aligned}
\mathrm{d}W_t =& \left(\tilde{a}_t \frac{\delta}{N}(\omega - \tilde{a}_t) + \frac{\delta}{2N}(\omega - \tilde{a}_t)^2\right)\mathrm{d}N_t^B + \left(-\tilde{b}_t \frac{\delta}{N}(\tilde{b}_t + \omega) + \frac{\delta}{2N}(\tilde{b}_t + \omega)^2\right)\mathrm{d}N_t^S + 1_{\{t\le d\}}\omega\frac{f}{N}\mathrm{d}t \\
=& \frac{\delta}{2N}\left(\omega^2 - (\tilde{a}_t)^2\right)\mathrm{d}N_t^B + \frac{\delta}{2N}\left(\omega^2 - (\tilde{b}_t)^2\right)\mathrm{d}N_t^S + 1_{\{t\le d\}}\omega\frac{f}{N}\mathrm{d}t, \quad \forall t > 0,
\end{aligned}
$$

where $\tilde{a}_t$ and $\tilde{b}_t$ are ask and bid price pressures set by the social planner at time t. And $\frac{1}{N}$-th of the inventory is

$$
\mathrm{d}i_t = \frac{f}{N}\mathrm{d}t - \frac{\delta}{N}(\omega - \tilde{a}_t)\mathrm{d}N_t^B + \tilde{b}_t\frac{\delta}{N}(\tilde{b}_t + \omega)\mathrm{d}N_t^S.
$$

Let us first suppose there is no liquidation, i.e., $f = 0$. In this case, the $1/N$-th welfare is independent of time t, let us denote it by

$$
v^S(i) = \sup_{\tilde{a},\tilde{b}} \mathbb{E}\left[\int_0^\infty e^{-\beta s}\left(\mathrm{d}W_t - \eta(i_t)^2\right)\middle| i_0 = i\right].
$$

By the dynamic programming principle, we know that v solves the HJB equation

$$
\begin{aligned}
-\eta i^2 - \beta v^S + \lambda \sup_{\tilde{a},\tilde{b}}\Bigg[&\frac{\delta}{2N}\left(\omega^2 - (\tilde{a})^2\right) + v^S(i - \frac{\delta}{N}(\omega - \tilde{a})) - v^S(i) \\
&+ \frac{\delta}{2N}\left(\omega^2 - (\tilde{b})^2\right) + v^S(i + \frac{\delta}{N}(\tilde{b} + \omega)) - v^S(i)\Bigg] = 0.
\end{aligned}
$$

Following the argument used in the proof of Proposition 1, we obtain that

$$
v^S(i) = E_\theta^S - F_\theta^S i^2,
$$

where

$$
E_\theta^S = \frac{\lambda}{\beta}\frac{\delta\omega^2}{N + 2\delta A_S}, \quad F_\theta^S = A_S,
$$

with A_S being the unique position solution to the equation

$$
\eta - \beta A = \frac{4\delta\lambda A^2}{N + 2\delta A}. \tag{80}
$$

More specifically,

$$A_S \equiv A_S(N) = \frac{2\delta\eta - N\beta + \sqrt{16N\delta\lambda\eta + 4\delta^2\eta^2 + 4N\delta\eta\beta + N^2\beta^2}}{4\delta(\beta + 2\lambda)} > 0. \tag{81}$$

The price pressures are given by $\tilde{a}_t = \tilde{a}(i_t)$, $\tilde{b}_t = \tilde{b}(i_t)$, with price policy functions given by

$$\tilde{a}(i) = A_\theta^S - B_\theta^S N i, \tag{82}$$

$$\tilde{b}(i) = - A_\theta^S - B_\theta^S N i, \tag{83}$$

where

$$A_\theta^S = \frac{2\delta A_S}{N + 2\delta A_S}\omega, \quad B_\theta^S = \frac{2A_S}{N + 2\delta A_S}.$$

Next we prove that $A_S < A^\star$. Let us consider positive, increasing functions $f_1(A) = \frac{8\delta\lambda A^2(1+\delta A)}{(N+1+2\delta A)^2}$, $f_2(A) = \frac{4\delta\lambda A^2}{N+2\delta A}$ for $A > 0$, then recall that $A^\star$ and A_S are uniquely determined by

$$\eta - \beta A - f_1(A) \begin{cases} > 0, & \text{if } 0 < A < A^\star, \\ < 0, & \text{if } A > A^\star, \end{cases} \quad \eta - \beta A - f_2(A) \begin{cases} > 0, & \text{if } 0 < A < A_S, \\ < 0, & \text{if } A > A_S. \end{cases}$$

Notice that for any $A > 0$

$$0 < \frac{f_1(A)}{f_2(A)} = \frac{2(1 + \delta A)}{(N + 1 + 2\delta A)} \frac{N + 2\delta A}{N + 1 + 2\delta A} < 1,$$

so

$$\eta - \beta A^\star - f_2(A^\star) < \eta - \beta A^\star - f_1(A^\star) = 0,$$

which implies that $A^\star > A_S$.

To prove the claim about the bid-ask spreads, notice that $\frac{4\delta A}{N+2\delta A}$ is strictly increasing for positive A, so the bid-ask spread set by the social planner ($2A_\theta^S = \frac{4\delta A_S}{N+2\delta A_S}\omega$) is strictly smaller than $\frac{4\delta A^\star}{N+2\delta A^\star}\omega$. On the other hand, recall that the bid-ask spread in the Stackelberg game of competition is at least $\frac{2(1+2\delta A^\star)}{N+1+2\delta A^\star}\omega$, therefore the difference between the bid-ask spread in the Stackelberg game and that set by the social planner is at least

$$[\frac{2(1 + 2\delta A^\star)}{N + 1 + 2\delta A^\star}\omega - \frac{4\delta A^\star}{N + 2\delta A^\star}\omega = \frac{2N}{(N + 1 + 2\delta A^\star)(N + 2\delta A^\star)}\omega > 0.$$

Hence, the social planner always sets a lower bid-ask spread.

The welfare and the price policy expressions given above are still valid for $t > d$. Under sunshine trading, let

$$v^{S☼}(i,f,t) = \sup_{\tilde{a},\tilde{b}} \mathbb{E}\left[\int_t^\infty e^{-\beta(s-t)}\left(\mathrm{d}W_t - \eta\, i_t^2\right)\middle|\, i_t = i\right].$$

Then $v^{S☼}(i,f,t) = v^S(i)$ if $t > d$. For $t \in [0,d]$, following the argument used in the proof of Proposition 2 we obtain that

$$v^{S☼}(i,f,t) = v^S(i) + 1_{\{t\le d\}}\left(I^{S☼}(d-t,f) + J^{S☼}(d-t)f\,i\right),$$

where

$$J^{S☼}(u) = -\frac{2A_S^2}{N\eta}\left(1 - e^{-\frac{\eta}{A_S}u}\right), \tag{84}$$

$$I^{S☼}(u,f) = f\frac{\omega}{N}\frac{1-e^{-\beta u}}{\beta} + f^2\int_0^u e^{-\beta(u-s)}J^{S☼}(s)\left(\frac{1}{N} + \frac{\delta\lambda J^{S☼}(s)}{(N+2\delta A_S)}\right)\mathrm{d}s. \tag{85}$$

Moreover, the optimal price pressures are given by $\tilde{a}_t = \tilde{a}^☼(i_t,f,t), \tilde{b}_t = \tilde{b}^☼(i_t,f,t)$, with price policy functions given by

$$\tilde{a}^☼(i,f,t) = \tilde{a}(i) - 1_{\{t\le d\}}f\left(G_\theta^{S☼}(d-t)\right),$$
$$\tilde{b}^☼(i,f,t) = \tilde{b}(i) - 1_{\{t\le d\}}f\left(G_\theta^{S☼}(d-t)\right),$$

where

$$G_\theta^{S☼}(u) = \frac{2A_S^2}{(N+2\delta A_S)\eta}\left(1 - e^{-\frac{\eta}{A_S}u}\right).$$

Thus, the bid-ask spread is not elevated during the liquidation period. Moreover, given the price policy and a zero initial inventory of market makers, the large seller's objective can be shown to be

$$\begin{aligned}
&\int_0^d e^{-\beta t}f\left(\mathbb{E}\left[\tilde{b}(i_t,f,t)\right] + \omega\right)\mathrm{d}t\\
=& f\frac{N\omega}{N+2cA_S}h_\beta(d) - f^2\frac{2A_S}{N+2cA_S}\left(\frac{1}{M_S}h_\beta(d) - \left(\frac{\beta A_S}{\eta}\frac{1}{M_S} + \frac{M_S A_S}{\eta}\frac{1}{M_S+\frac{\eta}{A_S}}e^{-\frac{\eta}{A_S}d}\right)\right.\\
&\left.\times h_{\frac{\eta}{A_S}}(d) - \frac{e^{-\beta d}}{M_S+\frac{\eta}{A_S}}h_{M_S}(d)\right).
\end{aligned} \tag{86}$$

where $M_S = \frac{4\delta\lambda A_S}{N+2\delta A_S}$ and $h_\gamma = \frac{1-e^{-\gamma u}}{\gamma}$ for a positive parameter $\gamma > 0$. It follows from (86) that the optimal sunshine liquidation intensity is given by

$$f^{\text{☼Planner}}(d) = \frac{N\omega}{4A_S}\left\{\frac{1}{M_S} - \left(\frac{\beta A_S}{\eta}\frac{1}{M_S} + \frac{M_S A_S}{\eta}\frac{1}{M_S+\frac{\eta}{A_S}}e^{-\frac{\eta}{A_S}d}\right)\frac{h_{M_S+\beta}(d)}{h_\beta(d)} - \frac{e^{-\beta d}}{M_S+\frac{\eta}{A_S}}\frac{h_{M_S}(d)}{h_\beta(d)}\right\}^{-1}.$$

Following the calculation in the proof of Lemma 5, it can be shown that $f^{\text{☼Planner}}(d)$ is strictly decreasing, and

$$\lim_{d\downarrow 0} f^{\text{☼Planner}}(d) = \infty.$$

The case of stealth trading can be treated similarly, and we obtain that the total expected welfare is given by

$$v^{S\text{☁}}(i,f,t) = v^S(i) + 1_{\{t\le d\}}\left(I_\theta^{S\text{☁}}(f) + J_\theta^{S\text{☁}} f\, i\right),$$

where

$$J_\theta^{S\text{☁}} = -\frac{2A_S}{N}\frac{A_S}{\nu A_S+\eta}, \tag{87}$$

$$I_\theta^{S\text{☁}}(f) = f\frac{\omega}{N}\frac{1}{\nu+\beta} + f^2\frac{1}{\nu+\beta}J_\theta^{S\text{☁}}\left(\frac{1}{N} + \frac{\delta\lambda J_\theta^{S\text{☁}}}{(N+2\delta A_S)}\right). \tag{88}$$

The optimal price pressures are given by $\tilde a_t = \tilde a^{\text{☁}}(i_t,f,t), \tilde b_t = \tilde b^{\text{☁}}(i_t,f,t)$, with price policy functions given by

$$\tilde a^{\text{☁}}(i,f,t) = \tilde a(i) - 1_{\{t\le d\}} f\left(G_\theta^{S\text{☁}}\right),$$
$$\tilde b^{\text{☁}}(i,f,t) = \tilde b(i) - 1_{\{t\le d\}} f\left(G_\theta^{S\text{☁}}\right),$$

where

$$G_\theta^{S\text{☁}} = \frac{2A_S}{N+2\delta A_S}\frac{A_S}{\nu A_S+\eta}.$$

Thus, the bid-ask spread is also not elevated during liquidation. □

Proof of Lemma 8. We need to prove the positivity of $I_\theta^{\text{☁}}(f)$ given in (63). We notice from (63) that for any given $f>0$, as ν becomes large, $I_\theta^{\text{☁}}(f)$ is essentially an average of $I_\theta^{\text{☼}}(u,f)$ for small $u>0$. By Lemma 1, we know that $I_\theta^{\text{☼}}(u,f)>0$ for all $u>0$ sufficiently small, hence $I_\theta^{\text{☁}}(f)>0$ for any $f>0$ if $\nu>0$ is sufficiently large.

Moreover, if (48) holds, then by Lemma 1 we deduce that $I_\theta^{\text{☼}}(u,f)>0$ for any $u>0$ and $f>0$, so $I_\theta^{\text{☁}}(f)>0$ for any $f>0$. □

Proof of Lemma 9. Following the same argument as in the proof of Lemma 2, one can obtain that $\mathbb{E}[i_{jt}] = f \times g(t)$, where

$$\tilde{g}(t) = \begin{cases} \dfrac{N + 2\delta A^\star}{N} \dfrac{1}{4\delta\lambda A^\star} \dfrac{\nu + \beta}{\nu + \frac{\eta}{A^\star}} (1 - e^{-Mt}), & t \leq d, \\ g(d) e^{-M(t-d)}, & t > d, \end{cases}$$

where $M = \frac{4\delta\lambda A^\star}{N+1+2\delta A^\star}$. This completes the proof. □

References

Admati, Anat and Paul Pfleiderer. 1988. "A Theory of Intraday Trading Patterns." *Review of Financial Studies* 1:3–40.

Almgren, Robert and Neil Chriss. 2001. "Optimal Execution of Portfolio Transactions." *Journal of Risk* 3:5–39.

Amihud, Yakov and Haim Mendelson. 1980. "Dealership Market: Market-making with Inventory." *Journal of Financial Economics* 8:31–53.

Amihud, Yakov, Haim Mendelson, and Lasse H. Pedersen. 2005. "Liquidity and Asset Prices." *Foundations and Trends in Finance* 1:269—364.

Bank, Peter, Ibrahim Ekren, and Johannes Muhle-Karbe. 2018. "Liquidity in Competitive Dealer Markets." *Working Paper, Carnegie Mellon University* .

Bertsimas, Dimitris and Andrew W. Lo. 1998. "Optimal Control of Execution Costs." *Journal of Financial Markets* 1:1–50.

Boulatov, Alex, Dan Bernhardt, and Ivan Larionov. 2016. "Predatory and Defensive Trading in a Dynamic Model of Optimal Execution by Multiple Traders." Manuscript, Higher School of Economics (Moscow).

Brough, Tyler Jon. 2010. "An Empirical Analysis of Institutional Liquidity Trading." Phd thesis, University of Arizona.

Brunnermeier, Markus and Lasse H. Pedersen. 2005. "Predatory Trading." *Journal of Finance* 60:1825–1863.

Choi, Jin Hyuk, Kasper Larsen, and Duane J. Seppi. 2018a. "Information and Trading Targets in a Dynamic Market Equilibrium." *Journal of Financial Economics, Forthcoming* .

———. 2018b. “Smart TWAP Trading in Continuous-Time Equilibria.” Manuscript, Carnegie-Mellon University.

Clark, Peter K. 1973. “A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices.” *Econometrica* 41:135–155.

Easley, David, Marcos M. López de Prado, and Maureen O’Hara. 2012. “The Volume Clock: Insights into the High-Frequency Paradigm.” *Journal of Portfolio Management* 39:19–29.

French, Kenneth R. 2008. “Presidential Address: The Cost of Active Investing.” *Journal of Finance* 63:1537–1573.

Gabaix, Xavier, Parameswaran Gopikrishnan, Vasiliki Plerou, and H. Eugene Stanley. 2006. “Institutional Investor and Stock Market Volatility.” *Quarterly Journal of Economics* 121:461–504.

Gatheral, Jim and Alexander Scheid. 2011. “Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework.” *International Journal of Theoretical and Applied Finance* 14:353–368.

Goettler, Ronald L., Christine Parlour, and Uday Rajan. 2005. “Equilibrium in a Dynamic Limit Order Market.” *Journal of Finance* 60:2149–2192.

Grinold, Richard C. and Ronald N. Kahn. 1999. *Active Portfolio Management: A Quantitative Approach for Producing Superior Returns and Controlling*. McGraw-Hill Education.

Grossman, Sanford and Merton Miller. 1988. “Liquidity and Market Structure.” *Journal of Finance* 43:617–633.

Hendershott, Terrence and Albert J. Menkveld. 2014. “Price Pressures.” *Journal of Financial Economics* 114:405–423.

Ho, Thomas and Stoll, Hans R. 1981. “Optimal Dealer Pricing under Transactions and Return Uncertainty.” *Journal of Financial Economics* 9:47–73.

Huberman, Gur and Werner Stanzl. 2005. “Optimal Liquidity Trading.” *Review of Finance* 9:165–200.

Investment Technology Group (ITG). 2007. “ITG ACE - Agency Cost Estimator: A Model Description.” Manuscript, ITG.

Jones, Charles. 2013. “What Do We Know About High-Frequency Trading?” *Columbia Business School Research Paper* 13-11.

Kyle, Albert S. 1985. “Continuous Auctions and Insider Trading.” *Econometrica* 53:1315–1335.

Kyle, Albert S. and Anna A. Obizhaeva. 2016. "Market Microstructure Invariance: Empirical Hypotheses." *Econometrica* 84:1345–1404.

Madhavan, Ananth N. 2000. "Market Microstructure: A Survey." *Journal of Financial Markets* 3:205—258.

van Kervel, Vincent, Amy Kwan, and Joakim Westerholm. 2018. "Order Splitting and Searching for a Counterparty." Manuscript, Pontificia Universidad Católica de Chile.

Menkveld, Albert. 2013. "High Frequency Trading and the New-Market Makers." *Journal of Financial Markets* 16:712–740.

———. 2016. "The Economics of High-Frequency Trading: Taking Stock." *Annual Review of Financial Economics* 8:1—24.

Obizhaeva, Anna A. and Jiang Wang. 2013. "Optimal Trading Strategy and Supply/Demand Dynamics." *Journal of Financial Markets* 16:1–32.

O'Hara, Maureen. 2015. "High Frequency Market Microstructure." *Journal of Financial Economics* 116:257–270.

Pritsker, Matthew. 2009. "Large Investors: Implications for Equilibrium Asset Returns, Shock Absorption, and Liquidity." Manuscript, Federal Reserve Board.

SEC. 2016. "Investment Company Liquidity Risk Management Programs." Release No. 33-10233; IC-32315. File No. S7-16-15.

Vayanos, Dimitri. 2001. "Strategic Trading in a Dynamic Noisy Market." *Journal of Finance* 56:131–171.

Vayanos, Dimitri and Jiang Wang. 2013. "Market Liquidity: Theory and Empirical Evidence." In *Handbook of the Economics of Finance*, edited by George Constantinides, Milton Harris, and René Stulz. Amsterdam: Elsevier Press.

Weill, Pierre-Oliver. 2007. "Leaning Against the Wind." *Review of Economic Studies* 74:1329–1354.

Zarinelli, Elia, Michele Treccani, J. Doyne Farmer, and Fabrizio Lillo. 2015. "Beyond the Square Root: Evidence for Logarithmic Dependence of Market Impact on Size and Participation Rate." *Market Microstructure and Liquidity* 1:1–32.