# WHAT DRIVES PRICE DISPERSION AND MARKET FRAGMENTATION ACROSS U.S. STOCK EXCHANGES?*

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We propose a theoretical model to explain two salient features of the U.S. stock exchange industry: (i) sizable dispersion and frequent changes in stock exchange fees; and (ii) the proliferation of stock exchanges offering identical transaction services, highlighting the role of *discrete pricing*. Exchange operators in the United States compete for order flow by setting "make" fees for limit orders ("makers") and "take" fees for market orders ("takers"). When traders can quote continuous prices, the manner in which operators divide the total fee between makers and takers is irrelevant because traders can choose prices that perfectly counteract any fee division. If such is the case, order flow consolidates on the exchange with the lowest total fee. The one-cent minimum tick size imposed by the U.S. Securities and Exchange Commission's Rule 612(c) of Regulation National Market Systems for traders prevents perfect neutralization and eliminates mutually agreeable trades at price levels within a tick. These frictions (i) create both scope and incentive for an operator to establish multiple exchanges that differ in fee structure in order to engage in second-degree price discrimination; and (ii) lead to mixed-strategy equilibria with positive profits for competing operators, rather than to zero-fee, zero-profit Bertrand equilibrium. Policy proposals that require exchanges to charge one side only or to divide the total fee equally between the two sides would lead to zero make and take fees, but the welfare effects of these two proposals are mixed under tick size constraints.

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# I. INTRODUCTION

Currently, stock prices are determined in stock exchanges through interactions between buyers and sellers. All stock exchanges in the U.S. are for-profit institutions that charge fees for transactions. In most finance models, however, stock exchanges either have no explicit role or make no profits in equilibrium. How do stock exchanges set their service fees? How does fee competition among stock exchanges shape the organization of the industry? In this paper, we propose a theoretical model to investigate the role of discrete pricing in creating two salient features of the U.S. stock exchange industry.

**Violation of the "law of one price."** Figure I shows the fee structures in ten major U.S. stock exchanges in May 2015. Fees differ across competing exchanges as well as across exchanges owned by the same holding company (hereafter "operator"). Frequent fee changes add to the complexity, as "the pressure to establish novel and competitive pricing often leads exchanges to modify their pricing frequently, typically on a calendar-month basis" (U.S. Security and Exchanges Commission (SEC) 2015, p. 21). Such spatial and temporal dispersion of prices can hardly be justified by physical product differentiation, as these exchanges are so similar that the SEC even refers to some of them as "cloned markets" (SEC 2015, p. 22). All stock exchanges in the United States are organized electronic limit-order markets and a stock can be traded on any of them.[1] A trader can act as a liquidity maker by posting a limit order with a specified price and quantity. A trade occurs once another trader (a liquidity taker) accepts the terms of a previously posted limit order through a market order. Upon execution, the exchanges charge a "make" fee and a "take" fee per share to each side of the transaction, the sum of which, the so-called "total" fee, is a major source of exchanges' revenues.[2]

**[Insert Figure I about here]**

**Market fragmentation.** Another puzzle is the proliferation of stock exchanges offering almost identical services. Figure I demonstrates the market fragments along two dimensions: (i) multiple operators co-exist; and (ii) each operator offers multiple stock exchanges. The principle of tax-neutrality asserts that, at a given tax level, it does not matter who—buyer or seller—is liable for the tax. Therefore, all traders would choose the exchange with the lowest

[1] Unlisted trading privileges in the U.S. allow stocks to be traded outside the listing venue. The NYSE is the only exchange that declines the opportunity to trade stocks listed on other markets.

[2] For example, BATS reported in its filing for an IPO in 2015 that about 70% of its revenues come from transaction fees (p. F4 on BATS S1 registration statement). O'Donoghue (2015) estimates that 34.7% of the NASDAQ's net income is from the fees.

total fee. This prediction raises the question why an operator establishes multiple exchanges to trade the same assets. Furthermore, competition over the total fee among operators should lead to Bertrand equilibrium, resulting in a zero total fee and zero profit, and leaving no room for market entry with any fixed cost to establish an exchange. Yet new entries are commonly observed. For example, on October 22, 2010, BATS created a new trading exchange, BATS Y, in addition to its existing BATS X exchange.

These puzzles have drawn attention from regulators, and a plan to ban the maker/taker pricing model is under discussion (SEC 2015). Yet relevant studies have provided limited theoretical understanding about what drives complex fee structures and the proliferation of exchanges that adopt them.

We show that one driving force behind price dispersion and market fragmentation is *the discrete tick size.*[3] In this paper, we consider a game among exchange operator(s), a continuum of liquidity makers and liquidity takers with heterogeneous valuations. When liquidity makers can quote continuous prices, they are able to neutralize the make/take fee allocations by adjusting their quotes. Then they always choose the exchange with the lowest total fee. As a result, no operator has incentives to offer multiple exchanges, and the competition over the total fee between operators leads to Bertrand equilibrium. Although these predictions are consistent with canonical economic principles, they are inconsistent with the stylized facts.

We next consider the case in which liquidity makers can propose only discrete trading prices. This setup is motivated by SEC Rule 612(c) of Regulation National Market Systems (NMS), which restricts the pricing increment to a minimum of $0.01 if the security is priced equal to or greater than $1.00 per share.[4] Although liquidity makers cannot quote sub-penny prices, the make/take fees are not subject to the tick size constraints. Stock exchanges can use make/take fees to effectively propose sub-penny transaction prices that cannot be neutralized by liquidity makers. As a result, the discrete tick size changes the nature of price competition between exchanges from one-sided (over the total fee) to two-sided (over the make fee and the take fee).

[3] We are aware of other drivers for market fragmentation, such as computer-based high frequency trading (HFT). The rise of the exchanges adopting a maker/taker pricing model accompanies the proliferation of computer-based trading. Fees are only a fraction of a cent. Such a small profit margin per trade makes it hard for humans to gather a reasonable amount of profit. But a computer-based algorithm makes such maneuvers feasible. There are trading algorithms specializing in market making, and profiting from the fees and algorithms aiming to minimize transaction costs.

[4] We discuss the exemptions from this rule in Section IX.

Non-neutrality creates product differentiation for otherwise identical exchanges. All else being equal, a liquidity maker prefers the exchange with a lower take fee because it increases the probability that a liquidity taker will accept her offer. Therefore, exchanges with lower take fees are of higher "quality" to liquidity makers. From the point of the view of a liquidity maker, an operator's choice between make and take fees is equivalent to simultaneous choices about the price of the execution service (the make fee) and the quality of the execution service (the take fee).

This product differentiation then facilitates second-degree price discrimination. Discrete price can force liquidity makers with heterogeneous valuations to propose the same limit-order price. The operator then can open multiple exchanges with differentiated make/take fees to screen liquidity makers. Liquidity makers with larger (smaller) gains from execution select the exchanges with higher (lower) prices and higher (lower) execution probabilities. Interestingly, we show that such second-degree price discrimination increases not only the operator's profit but also the welfare of both liquidity makers and liquidity takers, because new exchanges create more effective transaction prices.

Choosing price and quality simultaneously destroys Bertrand equilibrium as well as any pure-strategy equilibrium. Consider a simple case of duopoly operators, each opening one exchange. No pure-strategy equilibrium exists when any operator charges a positive total fee, because competing operators have incentives to undercut each other toward zero total fees. Surprisingly, no pure-strategy equilibrium exists even when both operators charge zero total fees. If one operator charges a zero total fee, the other operator has two types of profitable deviation. One type increases the charge to liquidity makers by $\varepsilon$ while decreasing the charge to liquidity takers by $\mu \cdot \varepsilon$ $(0 < \mu < 1)$. This deviation reduces gains from execution but increases execution probability, which attracts liquidity makers with high gains from execution. The other type of deviation decreases the charge to liquidity makers while increasing the charge to liquidity takers, which appeals to liquidity makers with low gains from execution. We also show the non-existence of pure-strategy equilibrium when we allow operators to choose the number of exchanges, because operators can implement the fee structure mentioned above by establishing new exchanges.

We then prove the existence of symmetrical mixed-strategy equilibrium, and we show that any mixed-strategy equilibrium entrails positive profits. As in Varian (1980), the fact that only mixed-strategy equilibrium exists rationalizes the spatial price dispersion (exchanges have different fees at the same time) and the temporal price dispersion (exchanges vary their fees over time). The driver of the mixed-strategy equilibrium in our paper, however, differs from that of the mixed strategy equilibrium in the one-dimensional price competition. When firms compete over one price, the violation of the law of one price is driven either by the cost to consumers to obtain the prices (Rosenthal 1980; Varian 1980; Burdett and Judd 1983) or by the cost to firms to advertise their prices (Butters 1977; Baye and Morgan 2001). Our model does not include frictions to react or transmit prices. The driver of the mixed-strategy equilibrium is the inability to rank prices uniquely in a two-dimensional space.

To the best of our knowledge, we are the first to rationalize exchange fee dispersion and frequent fee changes, as documented by O'Donoghue (2015) and Cardella, Hao, and Kalcheva (2015). Colliard and Foucault (2012) predict that competition between exchanges leads to zero total fees. Two other extant interpretations shed light on the existence of rebates (negative fees), but they cannot explain why trading does not consolidate on the market with the maximum rebate. The first is that rebates to liquidity makers encourage liquidity provision (Malinova and Park 2015; Cardella, Hao, and Kalcheva 2015), which in turn attracts liquidity takers. Yet this interpretation cannot explain why new entries to the exchange industry, such as Direct Edge A and BATS Y, charge positive fees to liquidity makers. The second interpretation argues that retail brokers have incentives to route customer orders to markets with the highest rebates, as brokers do not need to pass the rebates on to their customers (Angel, Harris, and Spatt 2010, 2013; Battalio, Corwin, and Jennings 2015). But this interpretation does not explain why exchanges do not simply copy the highest rebate set by their competitors.

Our paper also contributes to the literature on market fragmentation. The fragmentation of stocks trading has recently become a focus of research interest, because market fragmentation affects liquidity and price discovery (O'Hara and Ye 2011), leads to mechanical arbitrage opportunities for high frequency traders (Budish, Cramton, and Shim 2015; Foucault, Kozhan, and Tham 2015), and can serve as a driver of systemic risk, such as the "Flash Crash" of 2010 (Madhavan 2012). However, it is not clear why markets fragment in the first place, as the literature generally predicts consolidation of trading due to network externality or economies of scale (e.g., Stigler 1964; Pagano 1989; Chowdhry and Nanda 1991; Biais, Glosten, and Spatt 2005). The tick size channel in our paper explains market fragmentation both within the same operator and across operators. To our knowledge, no existing theoretical model predicts market fragmentation within the same operator. Foucault (2012) conjectures that the co-existence of various make/take fees on exchanges operated by the same operator should serve to screen investors by type. However, Foucault also mentions that "it is not clear however how the differentiation of make/take fees suffices to screen different types of investors since, in contrast to payments for order flow, liquidity rebates are usually not contingent on investors' characteristics (e.g., whether the investor is a retail investor or an institution)" (Foucault 2012, p. 20).[5] We address this puzzle: when end users cannot neutralize the breakdown of the total fee, the operators can screen liquidity makers based on the terms of trade offered to liquidity takers. Discrete tick size also provides a new interpretation of market fragmentation across operators. Researchers examining the co-existence of operators either assume exogenous operators, or rely on differentiated services or switching costs.[6] We contribute to the literature by showing that exchanges can endogenously co-exist in the absence of physical product differentiation, or frictions of search or switching.

Our modeling choice is inspired by the characteristics of stock markets, but its economic forces also shed light on other types of competition. Because SEC Rule 612(c) applies to displayed quotes, it is possible to create sub-penny pricing by designing either hidden order types or alternative trading systems (ATS), commonly referred to as "dark pools." In this sense, the existence of a tick size sheds light on the proliferation of new order types and dark pools. We also contribute to the burgeoning literature on two-sided markets. Rochet and Tirole (2006, p. 646) define two-sided markets as those in which "the volume of transactions between end-users depends on the structure and not only on the overall level of the fees charged by the platform." A fundamental question is whether the two-sided markets that Rochet and Tirole define can generate qualitatively different predictions from one-sided markets with identical setups.[7] We fill in this gap by showing that non-neutrality creates product differentiation among intrinsically homogeneous exchanges, which, in turn, creates price dispersion and leads to market fragmentation. Finally, Shaked and Sutton (1982) and Tirole (1988) predict the occurrence of non-Bertrand pure-strategy equilibrium when quality is chosen before price in a game. In our model, the "qualities" of the exchanges for liquidity makers are determined by the take fees, which can be adjusted as easily as the make fees. Such simultaneous choices of price and quality can lead to mixed-strategy equilibrium.

Besides rationalizing existing stylized facts pertaining to fee competition, our paper predicts the market outcomes of two alternative fee structures if policymakers were to ban the maker/taker pricing model: (i) charging fees only to one side; and (ii) distributing fees equally between two sides. We show that both proposals would reduce price competition to one

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[5] Since 2012, more exchanges have begun offering higher rebates to retail order flow, which can be another source of price discrimination.
[6] For models based on exogenous exchanges, see Glosten (1994), Parlour and Seppi (2003) and Foucault and Menkveld (2008). For product differentiation, see Pagnotta and Philippon (2015), Santos and Scheinkman (2001), Foucault and Parlour (2004), and Baldauf and Mollner (2015). For switching costs, see Cantillon and Yin (2008).
[7] For alternative definitions of two-sided markets, see Rysman (2009), Hagiu and Wright (2015), and Weyl (2010).

dimension, which could drive the make fee, the take fee, and the exchange's profit toward zero. Operators for traditional exchanges can survive short-run zero profit from make/take fees using other revenue sources, such as stock listings, but operators for the new exchanges do not have the same buffer. This provides one economic interpretation of why NYSE asks the regulator to ban the maker/taker pricing model,[8] but BATS aggressively opposes such a proposal.[9] Interestingly, we find that there are mixed welfare effects of banning the maker/taker pricing model. It could reduce welfare if liquidity makers' and liquidity takers' valuations were within the same tick, but increase welfare if their valuations were separated by price grids.

On April 5, 2012, Congress passed the Jumpstart Our Business Startups (JOBS) Act. Section 106 (b) of the act requires the SEC to examine the effect of tick size on initial public offerings (IPOs). A pilot program to increase tick size to five cents for small- and mid-cap stocks is to be implemented on May 15, 2016. Proponents of the proposal argue that a large tick size might increase market-making profit and support sell-side equity research and, eventually, increase the number of IPOs (Weild, Kim, and Newport 2012). We doubt the existence of such an economic channel. Even if it were to exist, we believe that a more direct consequence of increased nominal tick size would be more aggressive fee competition between exchanges to create effective price levels within the tick.

This article is organized as follows. In Section II, we describe the model. In Section III, we present the benchmark model in which the tick size is zero. In Section IV, we demonstrate the non-neutrality of the fee structure when there are tick size constraints. In Section V, we show product differentiation and liquidity makers' segmentation into multiple exchanges. In Section VI, we show price discrimination under monopoly. In Section VII, we derive the non-existence of pure-strategy equilibrium under competing operators. In Section VIII, we discuss the robustness of our model predictions. In Section IX, we discuss the policy implications of our model. We conclude in Section X and discuss the broader economic implications of our model. All proofs are presented in the Appendix.

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[8] See Jeffrey Sprecher, Chairman and Chief Executive Officer of Intercontinental Exchange, and owner of New York Stock Exchange, Statement to the U.S. Senate Banking, Housing and Urban Affairs Committee, Hearing on "The Role of Regulation in Shaping Equity Market Structure and Electronic Trading," on July 8, 2014, available at:http://www.banking.senate.gov/public/index.cfm?FuseAction=Hearings.Hearing&Hearing_ID=2e98337f-d5c5-490f-80e7-6c1c81af7243.

[9] See Joe Ratterman, Chief Executive Officer, and Chris Concannon, President, of BATS, "Open Letter to U.S. Securities Industry Participants Re: Market Structure Reform Discussion," on January 6, 2015, available at http://cdn.batstrading.com/resources/newsletters/OpenLetter010615.pdf.

## II. MODEL

We consider a three-period game between three types of risk-neutral players: exchange operator(s), a continuum of liquidity makers with valuation of a stock $v_b$ uniformly distributed on $[\frac{1}{2}, 1]$ and liquidity takers with valuation of the stock $v_s$ uniformly distributed on $\left[0, \frac{1}{2}\right]$. $v_b$ and $v_s$, respectively, are the liquidity makers' and liquidity takers' private information. Because liquidity makers have higher valuations than liquidity takers, the former intend to buy from the latter.[10]

We consider both monopoly and competition between exchange operators. Figure II demonstrates the timeline of the model. At Date 0, operators simultaneously choose the number of exchanges to establish and set make fee $f_m^i$ and take fee $f_t^i$ on each exchange $i$. A negative fee means a rebate to the trader. Fees are charged upon order execution. At Date 1, nature randomly draws one liquidity maker from the uniform distribution $[\frac{1}{2}, 1]$. The liquidity maker makes two decisions after observing the fee structures: (i) submitting a limit order of one share to exchange $i$ or to no exchange at all; and (ii) proposing the limit order at price $P^i$ if she chooses exchange $i$. At Date 2, nature randomly draws one liquidity taker from the uniform distribution $\left[0, \frac{1}{2}\right]$. The liquidity taker observes fees as well as the limit-order price proposed by the liquidity maker, and then decides whether to trade at $P^i$ on exchange $i$.[11] Once a trade happens, exchange $i$ profits from the total fee, $f_m^i + f_t^i$.

We determine the subgame-perfect equilibrium of the sequential-move game by backward induction. First, we look for the probability that a liquidity taker accepts the limit orders at price $P^i$ given the take fee $f_t^i$. Then, we study the liquidity maker's exchange choice and the optimal limit-order price $P^{i*}$ to maximize her expected surplus, given $f_t^i$ and $f_m^i$ and the liquidity taker's best response at Date 2. The liquidity maker can choose not to participate if her expected surplus is negative.[12] Finally, given the probability of participation based on the above subgames, a monopoly operator chooses the number of exchanges and the associated fee structures to maximize the expected profit, while competing operators choose the number

[10] When the liquidity maker intends to sell to the liquidity taker, our model predictions do not change, given that traders' valuations are symmetric and uniformly distributed.

[11] In practice, an exchange charges a fee to route a limit order to another exchange. This routing fee imposes a barrier to submitting an order to a low take-fee exchange aiming to interact with a limit order in a high take-fee exchange for the purpose of reducing the trading cost. For an example of a routing fee, see: https://www.batstrading.com/support/fee_schedule/edga/.

[12] We make a technical assumption that the liquidity maker cannot submit "stub quotes," that is, a limit-order price so low that no liquidity taker accepts it. Stub quotes lead mechanically to zero expected surplus. The SEC, however, prohibits market maker stub quotes. See https://www.sec.gov/news/press/2010/2010-216.htm.

of exchanges to establish and the associated fee structures simultaneously such that those strategies form a Nash equilibrium.

**[Insert Figure II about here]**

Our main purpose is to model exchange competition, so our model is parsimonious with respect to traders' choices between limit and market orders: traders do not choose the order type, and the limit-order book is empty when the liquidity maker arrives.[13] These two assumptions follow from Menkveld (2010) and Foucault, Kadan, and Kandel (2013).[14] Such simplification of the limit-order book allows us to gain economic insights that extend beyond the stock exchange industry; we discuss these implications in the Conclusion.

By proposing a nominal trading price $P^i$ to exchange $i$, the liquidity maker essentially chooses the cum fee buy and sell prices to be $p_b^i \equiv P^i + f_m^i$ and $p_s^i \equiv P^i - f_t^i$. Under zero tick size, $P^i$ can be any real number. Under a discrete tick size, $P^i$ can only be on price grids with the minimum distance as tick size; we call such restrictions tick size constraints. Our baseline model considers a tick size of 1, that is,

(1) $P^i \in \{n\}$, where *n* is an integer.

Under this assumption, the valuations of the liquidity maker and the liquidity taker fall within the same tick.

In Section VIII, we relax this assumption by allowing price to be any rational numbers with equal space, that is,

(1′) $\left\{P^i \in \frac{n}{N}\right\}$, where *n* is an integer and *N* is a natural number.

The tick size under (1′) is $\frac{1}{N}$. When $N > 1$, the support of traders' valuation goes beyond one tick.

Our main analysis focuses on $N = 1$, or tick size constraints (1). In Section VIII, we consider tick size constraints (1′) and demonstrate that our main results still hold for *N>1*.

[13] Theoretical studies on order-placing strategy generally provide a richer structure of order selection by assuming exogenous exchanges (Parlour and Seppi 2003; Foucault and Menkveld 2008; Rosu 2009). For example, Foucault and Menkveld (2008) find that two exogenous exchanges can co-exist because of queuing behavior. Our model, however, demonstrates the co-existence of endogenous trading exchanges.

[14] In practice, the decision between making and taking liquidity is not so rigid. Yet it is evident that traders specialize in trading activities, creating differentiation. For example, high-frequency traders account for a large fraction of liquidity supply in electronic markets (Menkveld 2010; Foucault, Kadan, and Kandel 2013). Our model captures this feature.

## III. BENCHMARK: NO TICK SIZE CONSTRAINTS

In this section, we analyze market outcomes when $P^i$ can be any real number. The results in this section serve as the benchmark for future discussions after we incorporate tick size constraints.

At Date 1, a liquidity maker proposes a limit-order price such that:

$$\max_{P^i}[(v_b - P^i - f_m^i) \cdot \Pr(v_s \leq P^i - f_t^i)]$$
$$= \max_{P_i}\left[\left(v_b - P^i - f_m^i\right) \cdot \max\left\{0, \min\{\, 2 \cdot \left(P^i - f_t^i\right), 1\}\right\}\right]$$

We denote the total fee on exchange $i$ as $T^i \equiv f_m^i + \, f_t^i$. The solution for the optimal limit-order price is

$$P^{i*}(v_b) = \frac{v_b + T^i}{2} - f_m^i, \text{ if } v_b \geq T^i. \tag{2}$$

Equation (2) shows that the nominal limit-order price strictly increases in the liquidity maker's valuation when $v_b \geq T^i$. The cum fee buy price and the cum fee sell price are $p_b^i(v_b) = P^i + f_m^i = \frac{v_b + T^i}{2}$ and $p_s^i(v_b) = P^i - f_t^i = \frac{v_b - T^i}{2}$, respectively. Therefore, the economic outcome is affected by the total fee $T^i$, and not by its breakdown into $f_m^i$ and $f_t^i$ under a zero tick size. In Proposition 1, we posit the market outcomes when the liquidity maker does not face tick size constraints.

**PROPOSITION 1** (market outcome under zero tick size). Under continuous pricing:

(i) only the total fee, not its breakdown, matters for the three types of players;

(ii) a monopoly operator has no incentive to open more than one exchange. The standing exchange charges a total fee of $\frac{3}{8}$ and earns a profit of $\frac{9}{64}$;

(iii) competing operators set total fees at zero and earn zero profits.

The neutrality of fee structures implies that all traders would rush to the exchange with the lowest total fee. As a result, the monopoly operator cannot benefit from opening more than one exchange. Also, competition among operators leads to Bertrand equilibrium. Colliard and Foucault (2012) show predictions similar to parts (i) and (iii) of Proposition 1.

Proposition 1 implies that we should not observe any price dispersion of total fees if the tick size is zero. We should also expect consolidation with one operator on one exchange, as long as establishing an exchange involves non-zero costs. Nevertheless, these predictions are not consistent with the stylized facts. Next, we demonstrate that tick size constraints can generate results that are dramatically different yet consistent with reality.

## IV. NON-NEUTRALITY OF FEE STRUCTURES UNDER TICK SIZE CONSTRAINTS

In this section, we show that the fee structure becomes non-neutral in the presence of tick size constraints (1). In Section IV.A. we establish the results for one exchange. In Section IV.B. we advance the non-neutrality intuition by showing that, with the same total fee but inverting make and take fees, one exchange can dominate the other.

### *IV.A. Fee Structure under Tick Size Constraints: One Exchange*

For a given limit-order price $P$, a trade occurs if and only if:

(3) $$\begin{cases} 0 \leq v_s \leq p_s = P - f_t \\ p_b = P + f_m \leq v_b \leq 1 \end{cases}$$

A necessary condition for the exchange to survive is a budget-balanced total fee, $T = f_m + f_t \geq 0$, which is equivalent to:

(4) $$P - f_t \leq P + f_m.$$

Thus, in order for a trade to occur, (3) and (4) together require:

(5) $$0 \leq v_s \leq P - f_t \leq P + f_m \leq v_b \leq 1.$$

For a trade to occur, the total fee set by the operator must be less than 1. Here we restrict our attention to $|f_i| \leq 1$ (*i=m,t*), because U.S. regulations prohibit any exchange from charging fees greater than one tick.[15] Lemma 1 establishes the result that the fee structure matters in the presence of tick size constraints.

**LEMMA 1** (fee structure and trading price under tick size constraints). Under tick size constraints (1) and $|\mathrm{f_i}| \leq 1$ (i=m,t).

(i) In order for a trade to happen and the exchange to survive, the exchange must charge one side while subsidizing the other. Moreover, the total fee cannot exceed the tick size. That is,

(6) $$\mathrm{f_m} \cdot \mathrm{f_t} < 0 \text{ and } 0 \leq \mathrm{f_m} + \mathrm{f_t} < 1.$$

*(ii)* The liquidity maker proposes a trading price*:*

[15] Allowing fees with absolute values greater than 1 does not change our results; we show in Proposition 5 that the liquidity maker can neutralize fees breakdowns that are multiples of a tick.

$$\text{(7)} \qquad P = \begin{cases} 0 & \text{when } f_m > 0 \text{ (so that } f_t < 0) \\ 1 & \text{when } f_m < 0 \text{ (so that } f_t > 0) \end{cases}.$$

(iii) The cum fee buy and sell prices are:

$$\text{(8)} \qquad \begin{aligned} p_b &\equiv P + f_m = \begin{cases} f_m & \text{when } f_m > 0 \\ 1 + f_m & \text{when } f_m < 0 \end{cases} \\ p_s &\equiv P - f_t = \begin{cases} -f_t & \text{when } f_m > 0 \\ 1 - f_t & \text{when } f_m < 0 \end{cases} \end{aligned}$$

When $f_m = f_t = 0$, Equation (5) cannot hold if $P$ must be an integer, except for the knife-edge cases of $v_b = 1$ or $v_s = 0$. It is also easy to see that Equation (5) cannot hold when both $f_m > 0$ and $f_t > 0$. Part (i) of Lemma 1 is thus established. Other parts of Lemma 1 follow directly from Equation (5).

Equation (6) shows that the make and take fees must carry opposite signs. This result is driven by the simplifying assumption that the liquidity maker's and liquidity taker's valuations are within the same tick.[16] Yet the prediction is consistent with the stylized facts. In reality, it is rare for a major exchange to charge both liquidity makers and liquidity takers positive fees. Among 108 fee structure changes documented by Cardella, Hao, and Kalcheva (2015), exchanges differ on which side they subsidize, but no exchange ever charges both sides positive fees.[17] Lemma 1 thus provides the first plausible explanation of this puzzle. Fees of opposite sign are able to create sub-tick cum fee buy and sell prices when the liquidity maker's and liquidity taker's valuations fall within the same tick.

The non-neutrality of the fee structure can be seen from Equation (8): cum fee buy and sell prices are now functions of $(f_m, f_t)$. In Section VIII, we show that such non-neutrality holds as long as the tick size is greater than zero. We start with a tick size of 1 to simplify the model, as it restricts the quotes proposed by a liquidity maker to either zero or 1, so that exchanges can mandate cum fee buy price $p_b$ and cum fee sell price $p_s$. When we reduce the tick size to less than 1 in Section VIII, exchanges can no longer mandate the cum fee buy and sell prices, because a liquidity maker has more price levels at which to post limit orders. Yet a tick size of 1 conveys the model's economic mechanism in the simplest way, and we show in

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[16] In Section VIII, we show that the make and take fees can both be positive when the liquidity maker's and liquidity taker's valuations range beyond one tick.

[17] We thank Laura Cardella for helping us confirm this claim. IEX, an Alternative Trading System currently applying for registration to become a national securities exchange, charges positive fees to both sides based on the market fairness argument. IEX is very far from achieving significant trading volume. It would be interesting to see whether IEX will change its fee structure if it ever becomes a national exchange subject to the same regulation as other exchanges are.

Section VIII that similar intuitions hold for any discrete tick size with much greater mathematical complexity. In addition, most studies of two-sided platforms assume that they can mandate the prices of both sides (Weyl 2010). Assuming a large tick size of 1 makes our pricing structure similar to other two-sided market models.

### *IV.B. The Maker/Taker Market vs. the Taker/Maker Market*

In this subsection, we consider competition between an exchange that subsidizes the liquidity maker while charging the liquidity taker (a maker/taker exchange) and an exchange that subsidizes the liquidity taker while charging the liquidity maker (a taker/maker exchange, also called an inverted fee exchange). In our game, the liquidity taker seems to play a passive role: she always follows the liquidity maker's exchange choice because the unselected exchange has an empty limit-order book. It thus seems that the priority of an exchange is to attract the liquidity maker, and that a natural way to attract the liquidity maker is to subsidize her, as the maker/taker market does. The literature is silent as to why the taker/maker market can attract the liquidity maker from a market that subsidizes her, particularly when the current U.S. regulation provides other advantages to subsidize the liquidity maker.[18, 19] We fill this gap by identifying two costs of a subsidy to a liquidity maker. Lemma 2 shows that the costs of such a subsidy can be so high that any liquidity maker prefers an exchange that charges her to an exchange that subsidizes her when the total fees in the two exchanges are the same.

**LEMMA 2.** Under tick size constraints (1), suppose exchange 1 adopts fee structure $\mathrm{F}^1 = (\alpha, \beta)$, and exchange 2 adopts fee structure $\mathrm{F}^2 = (\beta, \alpha)$, where $1 > \alpha > -\beta > 0$. The liquidity maker is indifferent to the two exchanges when $|\alpha| + |\beta| = 1$. She prefers exchange 1 when $|\alpha| + |\beta| < 1$, and exchange 2 when $|\alpha| + |\beta| > 1$.

Lemma 2 demonstrates that the liquidity maker prefers a market that charges her and subsidizes the liquidity taker when the tick size is large relative to the level of the make/take fees. Such a result is driven by two costs of a subsidy. First, a subsidy for a liquidity maker forces the liquidity maker to quote a more aggressive price: she proposes a limit order at $P = 0$

[18] Foucault, Kadan, and Kandel (2013) demonstrate that a monopoly exchange may choose to subsidize liquidity takers to maximize the trading rate of the exchange.

[19] The Reg NMS no trade-through rule requires orders to be routed to the exchange with the best nominal limit-order price. Colliard and Foucault (2012) show that liquidity makers are able to display better nominal prices when they obtain rebates, which encourages exchanges to give them such rebates. Battalio, Corwin, and Jennings (2015) find that retail brokers have incentives to route customer limit orders to exchanges with maximum rebates, because the regulation does not require them to pass rebates on to customers.

when charged, and a limit order at $P = 1$ when subsidized (Part (ii) of Lemma 1). Second, the subsidy of a liquidity maker comes from the charge to a liquidity taker. A higher take fee can reduce the probability that the liquidity taker accepts the limit order and thus the liquidity maker's probability to realize the gains from execution. Lemma 2 shows that the costs of a subsidy are higher when the tick size is larger relative to the sum of the absolute values of the make and take fees. Therefore, an increase in the tick size while holding the fee fixed would lead the liquidity maker to choose an exchange that charges her instead of subsidizing her.

## V. VERTICAL PRODUCT DIFFERENTIATION AND LIQUIDITY MAKERS' SEGMENTATION

In this section, we demonstrate that the non-neutrality of the fee structure, led by the tick size constraints, allows operators to create product differentiations for otherwise identical exchanges. Hereafter, we consider each exchange's decision variables as cum fee buy and sell prices to avoid a tedious discussion of fee structures that achieve the same equilibrium outcome.

### *V.A. Product Differentiation*

Given the cum fee sell price, the marginal liquidity taker's valuation on exchange *i* is:

(9) $$\hat{v}_s^i \equiv \min\left\{p_s^{\mathrm{i}}, \frac{1}{2}\right\}.$$

It follows that the probability that a liquidity taker accepts the buy limit order is:

(10) $$q_s^{\mathrm{i}} = \Pr\left(v_s \leq \hat{v}_s^i\right) = 2 \cdot \hat{v}_s^i = 2 \cdot \min\left\{p_s^{\mathrm{i}}, \frac{1}{2}\right\}.$$

Clearly, a higher cum fee sell price implies higher execution probability for the liquidity maker.

The liquidity maker's expected surplus for choosing exchange *i* is:

(11) $$BS^{\mathrm{i}} = \left(v_b - p_b^{\mathrm{i}}\right) \cdot q_s^{\mathrm{i}} = 2 \cdot \left(v_b - p_b^{\mathrm{i}}\right) \cdot \min\left\{p_s^{\mathrm{i}}, \frac{1}{2}\right\}.$$

Equation (11) shows that a liquidity maker's expected surplus increases with the cum fee sell price. All else being equal, a liquidity maker prefers an exchange with a higher cum fee sell price because such an exchange increases the probability of realizing gains from execution. Thus, two exchanges with differentiated cum fee sell prices are vertically differentiated for the liquidity maker: the exchange with the higher cum fee sell price is of higher quality. Such vertical product differentiation is the fundamental rationale behind the second-degree price discrimination we will discuss in Section VI.

In our model, exchanges *simultaneously* set the price and quality of their execution service for the liquidity maker. In the standard literature the decision about product quality is usually made *prior* to deciding product price, because product quality involves a long-term decision and price a short-term decision (Tirole 1988). Models with sequential quality and price choices usually predict non-Bertrand pure-strategy equilibrium (Shaked and Sutton 1982; Tirole 1988). Yet the "quality"—the execution probability—in our model is determined entirely by the cum fee sell price, which can be adjusted as easily as the cum fee buy price. Such a simultaneous choice is the main cause of the non-existence of pure strategy, as we will discuss in Section VII.

The differentiation across exchanges depends crucially on the non-neutrality of the fee structure. If end users can neutralize the fees set by an operator, exchanges are homogeneous and compete along one dimension, the total fee, as shown in Section III. Once end users cannot neutralize the fees, price competition becomes two-dimensional and an operator is able to set the price and quality of the product simultaneously.

### *V.B. Liquidity Makers' Segmentation*

Our model has only one liquidity maker, but that liquidity maker is drawn randomly from a uniform distribution. In this subsection, we consider the segmentation of the whole distribution of liquidity makers.

Given the cum fee buy and sell prices on exchanges 1 and 2, $(p_b^1, p_s^1)$ and $(p_b^2, p_s^2)$ respectively, a liquidity maker's expected surpluses for choosing exchange i is:

(12) $$BS^{\mathrm{i}} = \left(v_b - p_b^{\mathrm{i}}\right) \cdot q_s^{\mathrm{i}} = 2 \cdot \left(v_b - p_b^{\mathrm{i}}\right) \cdot p_s^i, \qquad i = 1,2$$

These equalities follow from $\hat{v}_s^i = p_s^i (i = 1,2)$, because neither exchange would set $p_s^i > \frac{1}{2}$): doing so would reduce its per-trade profit without increasing trading volume.

When $p_s^1 = p_s^2$, a liquidity maker's surplus would be $BS^1 \gtreqless BS^2$ if and only if $p_b^1 \lesseqgtr p_b^2$. Without loss of generality, suppose that $p_s^1 < p_s^2$, which implies that exchange 1 is of low quality and exchange 2 is of high quality. The liquidity maker's surpluses in each exchange are shown in Figure III.

**[Insert Figure III about here]**

When $p_b^1 \geq p_b^2$, as shown in Panel (a) of Figure III, $BS^1 \leq BS^2$ for any $v_b \geq p_b^2$. So any liquidity maker chooses exchange 2, because exchange 2 offers higher execution probability, along with a lower cum fee buy price.

When $p_b^1 < p_b^2$, there is a unique intersection:

(13) $$\varphi \equiv \frac{p_b^2 \cdot p_s^2 - p_b^1 \cdot p_s^1}{p_s^2 - p_s^1}$$

and $BS^1 \lesseqgtr BS^2$ for any $v_b \gtreqless \varphi$, as shown in Panel (b) of Figure III. Liquidity makers with a valuation higher than $\varphi$ choose high-quality exchange 2 and liquidity makers with a valuation lower than $\varphi$ choose low-quality exchange 1. Because we assume that $v_b$ is uniformly distributed on $[\frac{1}{2}, 1]$, exchanges 1 and 2 co-exist when $\frac{1}{2} < \varphi < 1$. All things being equal, liquidity makers prefer the high-quality exchange. Yet liquidity makers are not uniformly inclined to choose the higher execution probability. Those with larger gains from execution care more about execution probability than those with smaller gains. This heterogeneity across traders allows an exchange with higher cum fee buy and sell prices to coexist with an exchange with lower cum fee buy and sell prices.

## VI. MONOPOLY: PRICE DISCRIMINATION

In this section, we characterize a monopoly operator's optimal choice of the number of exchanges to offer as well as her choice of fee structure on each exchange. The purpose is to explore the second-degree price discrimination facilitated by product differentiation.

**PROPOSITION 2** (number of exchanges established by a monopoly operator and associated fee structures). Under tick size constraints (1), for a monopoly operator who operates k exchanges, the optimal cum fee buy and cum fee sell prices in each exchange *i* are:

(14) $$p_b^i = \frac{1}{2} + \frac{i}{2(2k+1)}, p_s^i = \frac{i}{2k+1} \text{ with } 1 \le i \le k.$$

A liquidity maker with valuation $v_b \in [\varphi_{i-1}, \varphi_i]$ chooses exchange $i$, where:

$$\varphi^i = \begin{cases} p_b^1 & \text{for } i = 0 \\ \dfrac{p_b^{i+1} p_s^{i+1} - p_b^i p_s^i}{p_s^{i+1} - p_s^i} & \text{for } 1 \le i \le k-1 \\ 1 & \text{for } i = k. \end{cases}$$

The liquidity maker's expected surplus, the liquidity taker's expected surplus, and the monopoly operator's expected profit are:

(15) $$BS^M(k) = \frac{k \cdot (k+1)}{3(2k+1)^2}, SS^M(k) = \frac{k \cdot (k+1)}{3(2k+1)^2}, \pi^M(k) = \frac{2k \cdot (k+1)}{3(2k+1)^2},$$

respectively, which all increase in *k*.

If opening a new exchange requires a fixed cost *c*, the number of exchanges opened by the monopoly operator is:

$$\bar{k} = max\left\{k \in \mathbb{N} \middle| \frac{4k}{3(2k+1)^2(2k-1)^2} \geq c\right\}.$$

In Proposition 2, we show that the monopoly operator uses $k$ exchanges to segment the uniformly distributed liquidity makers into $k+1$ groups. The group with the lowest gains from execution does not participate. Among the rest of the $k$ groups, liquidity makers with higher gains from execution tend to choose exchanges with higher cum fee buy and sell prices. The monopoly operator's strategy follows standard menu pricing under second-degree price discrimination. Mussa and Rosen (1978) find, for example, that a monopoly firm can screen customers by offering a menu with a quality-differentiated spectrum of goods of the same generic type.

Our model provides two unique features compared with the standard menu-pricing model. First, the two exchanges in our model are physically identical, and the quality of the exchange is the take fee. Therefore, the exchanges use the terms of the trade offered to liquidity takers to screen liquidity makers. This finding explains the puzzle raised by Foucault (2012) that "it is not clear however how the differentiation of make/take fees suffices to screen different types of investors." Second, we discover the source of this price discrimination: non-neutrality led by the discrete price, because such price discrimination does not exist when liquidity makers and liquidity takers can neutralize the fee breakdown.

We find that the monopoly operator's profit increases with the number of exchanges, but the marginal benefit of adding an exchange decreases with the number of existing exchanges. Any fixed cost for establishing exchanges thus constrains the number of exchanges.

Interestingly, Equation (15) indicates that the liquidity maker's and the liquidity taker's expected surpluses also increase with the number of exchanges. The welfare gain originates from a higher participation rate: increasing the number of exchanges creates more cum fee price levels within the tick. As $k$ goes to infinity, the lowest cum fee buy price across all exchanges, $\frac{k+1}{2k+1}$, approaches $\frac{1}{2}$, which indicates almost full participation by liquidity makers. In our model, the creation of new cum fee price levels reduces the inefficiency created by discrete tick size, which increases the expected trading gains for all parties.

Table I provides an example of second-degree price discrimination using make/take fees. Column (1) shows that a monopoly that operates one exchange sets the cum fee buy price at $p_b^M = \frac{2}{3}$ and the cum fee sell price at $p_s^M = \frac{1}{3}$. Liquidity makers with valuation in $[\frac{2}{3}, 1]$ and liquidity takers with valuation in $[0, \frac{1}{3}]$ participate in this exchange. The operator has a profit of $\frac{4}{27}$. The second and third columns show that a monopoly can screen liquidity makers by setting up two exchanges. The low-quality exchange sets the cum buy price at $p_b^1 = \frac{3}{5}$ and the cum fee sell price at $p_s^1 = \frac{1}{5}$. The execution probability is $\frac{2}{5}$ on the low-quality exchange, which attracts liquidity makers with lower gains from execution ($v_b \in [\frac{3}{5}, \frac{4}{5}]$). The high-quality exchange sets the cum fee buy price at $p_b^2 = \frac{7}{10}$ and the cum fee sell price at $p_s^2 = \frac{2}{5}$. The execution probability is $\frac{4}{5}$ on the high-quality exchange, which attracts liquidity makers with higher gains from execution ($v_b \in [\frac{4}{5}, 1]$). This second-degree price discrimination increases the monopoly's profit from $\frac{4}{27}$ to $\frac{4}{25}$. The expected surplus for both liquidity makers and liquidity takers increases from $\frac{2}{27}$ to $\frac{2}{25}$. The expected trading volume increases from $\frac{12}{27}$ to $\frac{12}{25}$, which provides a justification for the expected welfare gains for all parties.

**[Insert Table I about here]**

## VII. COMPETITION: THE NON-EXISTENCE OF PURE-STRATEGY EQUILIBRIUM

In this section, we consider the case of two competing operators, each of whom establishes one exchange. In Section A, we establish the non-existence of the pure-strategy equilibrium under tick size constraints. In Section B, we show the existence of symmetric mixed-strategy equilibrium and that any mixed-strategy equilibrium leads to positive profit.

### *VII.A. No Pure-strategy Equilibrium*

In Proposition 3, we show that tick size constraints destroy not only Bertrand equilibrium but also *any* pure-strategy equilibrium.

**PROPOSITION 3** (no pure-strategy equilibrium). Under tick size constraints (1), there exists no pure-strategy equilibrium when two exchanges compete.

The detailed proof of this proposition is in the Appendix. Here we sketch the proof and the corresponding intuitions. The non-existence of pure-strategy equilibrium with unequal profits follows the intuition in the Bertrand game. The lower-profit exchange can increase its expected profit by undercutting the higher-profit exchange's cum fee buy price by $\varepsilon$ and mimicking its cum fee sell price.

Our model also does not have pure-strategy equilibrium entailing positive and equal profits. If identical profits are led by identical price structures, one exchange can increase its expected profit by undercutting its rival's cum fee buy price by $\varepsilon$ and mimicking its rival's cum fee sell price. The two-dimensional price competition also raises the possibility that two exchanges have different price structures but the same total profits.[20] Without loss of generality, suppose that, initially, exchange 2 has higher quality than exchange 1 ($p_s^2 > p_s^1 = a$). Figure III shows that exchange 2 must have a higher cum fee buy price than exchange 1 ($p_b^2 > p_b^1 = b$), and that liquidity makers with high valuations choose exchange 2 while liquidity makers with low valuation choose exchange 1. A profitable deviation for exchange 2 is reducing the cum fee sell price to $a$ and undercutting exchange 1's cum fee buy price by setting its new cum fee buy price to $b - \varepsilon$. This deviation allows exchange 2 to attract low-valuation liquidity makers with infinitesimal profit concession. In addition, high-valuation liquidity makers still choose exchange 2 because (i) they prefer $(b - \varepsilon, a)$ to $(b, a)$ and (ii) they choose to participate because the cum fee buy price $b - \varepsilon$ is lower than their valuation. When $\varepsilon$ is sufficiently small, the expected profit from retaining high-valuation liquidity makers is greater than the infinitesimal decrease in profit from low-valuation liquidity makers.[21]

Unlike the Bertrand game, no pure-strategy equilibrium exists here even if both operators make zero profit. Two possible scenarios lead to the zero-profit outcome: (i) at least one side of the market does not participate; or (ii) the cum fee buy and sell prices are equal. Suppose that both exchanges have zero participation rates; in that case at least one of the two operators can deviate profitably by facilitating some trades. Then we need only consider the case in which one exchange sets the cum fee buy price equal to the cum fee sell price. Without loss of generality, suppose that $p_b^1 = p_s^1$. The proof then involves two types of deviation.

First, we consider $p_b^1 = p_s^1 \geq \frac{1}{2}$. Then exchange 2 can have a profitable deviation by setting $p_b^2 = p_b^1 - \mu\varepsilon$, and $p_s^2 = p_s^1 - \varepsilon$ with $\varepsilon > 0$ and $0 < \mu < 1$. When $p_b^1 = p_s^1 > \frac{1}{2}$, exchange 2 reduces the cum fee sell price but not the execution probability, because any liquidity taker accepts $p_s^2 = \frac{1}{2}$. A lower cum fee buy price then attracts liquidity makers with a

[20] Numerical examples of such cases are available from the author on request. For a general discussion of asymmetry networks, see Ambrus and Argenziano (2009).

[21] Unlike the Bertrand game, the deviation to $(b - \varepsilon, a)$ does not attract all the original customers of exchanges 1 and 2. The participation probability of the liquidity taker decreases due to a drop in the cum fee sell price.

valuation above $p_b^2$ to exchange 2, and exchange 2 makes a positive profit. When $p_b^1 = p_s^1 = \frac{1}{2}$, such a deviation reduces the execution probability, but also reduces the cum fee buy price, which attracts liquidity makers with lower gains from execution; this is illustrated in Panel (a) of Figure IV. Second, we consider $p_b^1 = p_s^1 < \frac{1}{2}$. In this case the execution probability is less than 1. Panel (b) of Figure IV shows that exchange 2 can have a profitable deviation by setting $p_b^2 = p_b^1 + \varepsilon$ and $p_s^2 = p_s^1 + \mu \cdot \varepsilon$. Such a deviation increases the execution probability, and also increases the cum fee buy price, which attracts liquidity makers with higher gains from execution.

**[Insert Figure IV about here]**

Traditional price-quality games (Shaked and Sutton 1982; Tirole 1988) feature non-Bertrand pure-strategy equilibrium. An important cause of the non-existence of pure-strategy equilibrium in our model lies in the simultaneous choice of price and quality. If we allow the operator to choose the take fee in the first stage and the make fee in the second stage, the model generates standard non-Bertrand pure-strategy equilibrium (unreported for brevity). In most industries, it is natural to assume that the product quality decision is made prior to the product price decision because price can often be adjusted faster than quality (Tirole 1988). Yet in our model "quality," in terms of execution probability, is determined purely by the cum fee sell price. The cum fee sell price can be adjusted as easily as the cum fee buy price. Therefore, in our setting, it is reasonable to consider simultaneous price and quality competition rather than the sequential moves typical in the standard vertical differentiation literature.

### *VII.B. Mixed-strategy Equilibrium*

The non-existence of pure-strategy equilibrium further motivates us to investigate symmetric mixed-strategy equilibria. It is a daunting task to find analytical solutions for all possible mixed-strategy equilibria. Nevertheless, we are able to demonstrate the existence of a symmetric mixed-strategy equilibrium and prove that it always entails positive profit.

**PROPOSITION 4** (mixed-strategy equilibrium). Under tick size constraints (1):

(i) symmetric mixed-strategy equilibrium exists;

(ii) in equilibrium, both exchanges earn strictly positive profits.

We demonstrate the existence of symmetric mixed-strategy equilibrium using Theorem 6* of Dasgupta and Maskin (1986), which studies the mixed-strategy equilibrium existence problem in a discontinuous game. The proof of Proposition 4 shows that our game satisfies conditions specified in the theorem. Therefore, our model features symmetric mixed-strategy equilibrium.

Varian (1980) states that the mixed-strategy equilibrium justifies temporal price dispersion, or a price change over time. From this perspective, our paper provides the first theoretical justification for diverse fee structures across exchanges and their frequent changes. As indicated by an SEC statement, "these exchanges compete vigorously on price which leads to some rather complicated fee schedule that can change from month to month, making it a near full-time job to keep track of them all." [22] The driver of our mixed-strategy equilibrium, however, differs significantly from those in canonical one-dimensional mixed-strategy equilibrium. In the literature, one-dimensional mixed-strategy equilibrium generally involves frictions that prevent customers from finding the best price (Rosenthal 1980; Varian 1980; Burdett and Judd 1983), or cost to firms to transmit their prices (Butters 1977; Baye and Morgan 2001). For example, Rosenthal (1980) assumes loyal customers, while Varian (1980) assumes uninformed customers who are not aware of better prices. The incentive to exploit loyal or uninformed customers prevents firms from undercutting each other toward the marginal cost. In our model, the liquidity maker and liquidity taker fully optimize their choices, are fully aware of the fees, and exchanges do not feature any costs to transmit make/take fees to traders. The driving force behind the mixed-strategy equilibrium is two-dimensional price competition. When price competition is one-dimensional, all customers prefer a lower price even if their valuations are heterogeneous. No such consensus exists if price competition is two-dimensional. As Figure 3 demonstrates, liquidity makers choose different fee structures based on gains from execution.

The existence of strictly positive profit in mixed-strategy equilibrium rationalizes the entries of new exchanges and market fragmentation. This result arises from the two-sided feature of the markets caused by tick size regulation. When the tick size is zero, as shown in Section III, the markets are one-sided. Hence, competition between two exchanges can drive their profits to zero (Colliard and Foucault 2012), which implies that any positive cost involved in establishing a new stock exchange would deter entry. In reality, however, we continue to witness "the formation of new exchanges to experiment different price structures."[23] When the

[22] See Richard Holley, SEC Trading and Division, Statement in Panel Discussion at SEC Equality Market Structure Advisory Committee Meeting on October 27, 2015, available at https://www.sec.gov/news/otherwebcasts/2015/equity-market-structure-advisory-committee-102715.shtml.
[23] Holley, fn 22.

tick size is positive, the markets become two-sided. Competition between exchanges does not lead them to earn zero profit, which encourages the entry of new exchanges. Regulators are often concerned that the entry of new stock exchanges will generate greater market fragmentation (O'Hara and Ye 2011), but there is only limited understanding of why the market becomes increasingly fragmented. We show that one driving force behind fragmentation is the existing tick size regulation.

## VIII. ROBUSTNESS CHECK AND EXTENSIONS

In this section, we relax two assumptions made in Sections IV–VII. In Section VIII.A. we allow trades to occur on more price grids by reducing the tick size while keeping traders' valuations constant. In Section B, we relax the assumption of duopoly operators, and allow each operator to choose the number of exchanges. These extensions produce qualitatively similar results to those reported in Sections IV–VII.

### *VIII.A. Multiple Ticks*

In previous sections we compare market outcomes when the tick size is equal to zero with those that obtain when the tick size is equal to 1. All else being equal, in this subsection, we consider the case in which the tick size is equal to $\frac{1}{N}$, where $N$ is a natural number greater than 1. As a liquidity maker has more price grids from which to choose, exchanges cannot mandate unique cum fee buy and sell prices by setting make and take fees. Predictions under this relaxed setting, however, are qualitatively similar to those under $N = 1$. As long as the tick size is not zero, exchanges can use make and take fees to create sub-tick prices that cannot be neutralized by end users, which facilitates second-degree price discrimination and destroys pure-strategy equilibrium.

We start from the simplest case, in which a monopoly opens one exchange. We solve the problem backwards by first considering the limit-order price proposed by the liquidity maker.

**PROPOSITION 5** (the limit-order price proposed by the liquidity maker). Under tick size constraints (1′)

i) The liquidity maker submits no limit order when $v_b < f_m + \frac{[N \cdot f_t]}{N}$, and a liquidity maker submits a limit order at price $P = \frac{n(v_b, f_m, f_t)}{N}$ otherwise. Here

$$n(v_b, f_m, f_t) = \left\lfloor \frac{N \cdot (v_b - f_m + f_t) + 1}{2} \right\rfloor,$$

where $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the ceiling and floor functions, respectively.

ii) Compared with fee structure $(f_m, f_t)$, fee structure $(f_m + \frac{k}{N}, f_t - \frac{k}{N}), k \in Z$ shifts the limit-order price proposed by the liquidity maker by -k ticks, but leads to the same cum fee buy and sell prices as those that occur under fee structure $(f_m, f_t)$.

Proposition 5 shows that, for a given total fee, if the exchange increases its make fee by, say, one tick, then a liquidity maker can neutralize the effect of this increase by proposing a limit-order price that is one tick lower, leaving the cum fee buy and sell prices unchanged. Therefore, Part (i) of Proposition 1 is a limiting case of Proposition 5: when the tick size is zero, a liquidity maker is able to neutralize any fee breakdowns.

Proposition 5 also demonstrates that the limit-order price is a non-decreasing step function of the liquidity maker's valuation: a liquidity maker with higher gains from execution tends to propose a higher limit-order price to increase the probability of execution. Unlike continuous pricing, the limit-order price does not strictly increase in $v_b$, as $\frac{n(v_b, f_m, f_t)}{N}$ involves a floor function of $v_b$. When pricing is discrete, the unconstrained limit-order price that a liquidity maker would propose might not coincide with any of the price grids, which results in the same limit-order price proposed by liquidity makers with heterogeneous valuations.

The second-degree price discrimination under multiple ticks involves screening liquidity makers who have heterogeneous valuations but quote the same price. Obtaining the analytical solution for the optimal fee structure is a complex process, because the nominal price proposed by the liquidity maker $n(v_b, f_m, f_t)$ involves a floor function of $f_m$ and $f_t$. For the case of a monopoly opening one exchange, we are able to obtain the optimal fee structure as a function of *N*, and we present the results in Appendix B. We are not able to obtain a closed-form solution for the optimal make and take fees as a function of *N* when the operator establishes two exchanges. Therefore, we conduct a numerical search and present the results for *N* = 2, 4, and 8 as examples in Table II.[24]

[24] Our Internet Appendix provides C++ code for the simulations. We restrict the level of the take fee to $[-\frac{1}{N}, 0]$ in the simulation, as the equivalence of fee structures $(f_m, f_t)$ and $(f_m + \frac{k}{N}, f_t - \frac{k}{N}), k \in Z$ suggested by Proposition 5.

Table II shows that the $N$ = 2 case is identical to the $N$ = 1 case, as the tick size is still large enough for the monopoly operator to mandate unique cum fee buy and sell prices by setting make and take fees. The $N$ = 4 and $N$ = 8 cases illustrate two interesting features.

**[Insert Table II about here]**

First, the monopoly obtains strictly higher profits by establishing two vertically differentiated exchanges than by establishing one exchange. For example, for $N$ = 4, the low-quality exchange sets the take fee at −0.0340 and the high-quality exchange sets the take fee at −0.1595. The exchange with the lower take fee is of higher quality because a liquidity maker always chooses the exchange with the lower take fee if both exchanges set the same make fee.[25] The second-degree price discrimination involves charging a low make fee (0.4095) for the low-quality exchange and a high make fee (0.5840) for the high-quality exchange. The operator increases her total profit from 0.1406 to 0.1445.[26]

Second, liquidity makers rotate between the low-quality and the high-quality exchange as their valuation increases. For example, under tick size $\frac{1}{4}$, a liquidity maker with valuation $v_b \in [0.5000,0.6313]$ proposes a limit order at $P = 0$ on the low-quality exchange. A liquidity maker with valuation $v_b \in [0.6313, 0.7562]$ chooses the high-quality exchange, and proposes the same price $P = 0$. Interestingly, a further increase in valuation to $v_b \in [0.7562, 1.000]$ leads a liquidity maker to switch back to the low-quality exchange with an increased limit-order price of $P = \frac{1}{4}$. In fact, column (9) in Table II reveals that a liquidity maker with the lowest valuation $v_b \in [0.5000, 0.6501]$ chooses the high-quality exchange and proposes a limit order at price $P = 0$.

The rotation of the exchange choice seems to suggest horizontal differentiation of exchanges. Yet exchanges in our model are only vertically differentiated: all else being equal, all liquidity makers prefer an exchange with a lower take fee. Exchange rotation is driven by the definitions of the high valuation and low valuation types in our model. Unlike the usual models of vertical differentiation, the high and low types in our game are defined on each price

[25] To see this, consider two exchanges with different take fees but charging the same make fee. Suppose that the optimal limit-order price for the liquidity maker is $P^*$ on the high take-fee exchange. If the liquidity maker proposes the same $P^*$ on the low take-fee exchange, her gain from execution remains the same but the lower take fee increases her execution probability. Therefore, she would achieve a higher expected surplus by proposing the same $P^*$ on the lower take-fee exchange. The optimal price for the liquidity maker on the low-fee exchange may be different from $P^*$on the high-take fee exchange, but a different optimal price implies that the liquidity maker can achieve even higher expected surplus than proposing $P^*$ in the low take-fee exchange.

[26] 0.1445 is the sum of the profits from the low-quality exchange and the high-quality exchange.

grid. Table II shows that, at the same proposed limit-order price, liquidity makers who choose the high-quality exchange have higher valuation than those who choose the low-quality exchange. For example, for $N$ = 8, liquidity makers who propose $P = \frac{1}{8}$ on the low-quality exchange have valuations in [0.6501, 0.7334], whereas liquidity makers who propose $P = \frac{1}{8}$ on the high-quality exchange have valuations in [0.7334, 0.9164]. Yet a further increase in valuation beyond 0.9164 makes a liquidity maker propose a price of $\frac{1}{4}$ on the low-quality exchange. The nature of such second-degree price discrimination is to screen liquidity makers with relatively high and low valuations on the same price grid. Therefore, the price discrimination with multiple ticks serves as an extension of our baseline model with only one tick, in which all liquidity makers have to quote the same price.

### *VIII.B. Multiple Operators Each Choosing the Number of Exchanges*

In this subsection, we allow multiple operators to participate in the game, and each operator can choose the number of exchanges to offer. Still, no pure-strategy equilibrium exists as long as pricing is discrete.

**PROPOSITION 6**. Under tick size constraints (1′), no pure-strategy equilibrium exists when the number of operators is greater than 1.

Allowing an operator to choose the number of exchanges introduces a new type of profitable deviation: increasing the number of exchanges. Proposition 6 holds for any discrete tick size. Here we offer an intuitive explanation based on a tick size of 1. Suppose the number of exchanges with positive profits is $H$. We can always find an operator $i$ who does not own all these $H$ exchanges. Operator $i$ can increase her profits by establishing $H$ new exchanges, each of which undercuts the cum fee buy prices of the existing $H$ exchanges by $\varepsilon$, and sets the same cum fee sell prices. Such a deviation allows operator $i$ to capture the entire market. In reality, opening an additional exchange to compete with rivals is certainly more aggressive than changing fees on an existing exchange. Yet we find evidence consistent with this strategy. The exchange industry starts from the maker/taker model that offers rebates to liquidity makers while charging takers. On April 1, 2009, the Boston Stock Exchange became the first exchange to charge liquidity makers and subsidize liquidity takers. This inverted fee structure was subsequently adopted by Direct Edge’s new exchange EDGA, and BATS soon followed by establishing the BATS Y.[27] Among the three current major operators—NASDAQ OMX, Intercontinental Exchange (ICE), and BATS Global Markets)—only the ICE has no exchange with an inverted fee structure. However, in a recent panel discussion held by the SEC, the president of the NYSE admitted that, facing pressure from competitors, the NYSE is considering establishing an exchange with an inverted fee structure.[28]

Now consider the case in which all exchanges make zero profit. An exchange has zero profit if (i) no trader participates in that exchange or (ii) the cum fee buy price equals the cum fee sell price. For exchanges with a positive participation rate, we can find the one with the lowest cum fee price, $p^{min}$. Operator $i$ can find a profitable deviation by establishing a new exchange using the same deviating strategy depicted in the proof of Proposition 3. If $p^{min} < \frac{1}{2}$, the new exchange can choose $p_b^{inew} = p^{min} + \varepsilon$ and $p_s^{inew} = p^{min} + \mu\varepsilon$ $(0 < \mu < 1)$. This pricing structure increases both the cum fee buy price and execution probability, thus attracting liquidity makers with high gains from execution. If $p^{min} \geq \frac{1}{2}$, the new exchange can choose $p_b^{inew} = p^{min} - \mu\varepsilon$ and $p_s^{inew} = p^{min} - \varepsilon$ and attracts liquidity makers with low gains from execution.

## IX. DISCUSSION AND POLICY IMPLICATIONS

In this section, we discuss the policy implications of our results. In Section IX.A., we discuss recent policy debates on the maker/taker pricing model. In Section IX.B., we discuss the new insights our paper provides on the proliferation of new order types and alternative trading systems such as dark pools.

### *IX.A. Policy Debates on the Maker/Taker Pricing Model*

Recently, the holding company of the NYSE, the ICE, proposed eliminating the maker/taker pricing model.[29] Two replacement fee structures were proposed: (i) reducing take fees after eliminating rebates[30]; and (ii) distributing the total fee equally between liquidity

[27] BATS established BATS Y before its acquisition of Direct Edge.
[28] See Thomas Farley, President of the NYSE, panel discussion during the Equity Market Structure Advisory Committee Meeting at SEC on October 27, 2015, available at: https://www.sec.gov/news/otherwebcasts/2015/equity-market-structure-advisory-committee-102715.shtml.
[29] Sprecher, fn 8.
[30] Sprecher, fn 8.

makers and liquidity takers[31]. BATS, on the other hand, aggressively opposes banning the maker/taker pricing structure.[32] Proposition 7 predicts market outcomes under these two proposed fee structures.

**PROPOSITION 7**. Under tick size constraints (1′), eliminating rebates, thereby charging only one side, or splitting the total fee equally between a liquidity maker and a liquidity taker:

(i) leads competing operators to make zero profit;

(ii) discourages operators from opening more than one exchange.[33]

First, consider the policy proposal to remove rebates to liquidity makers and to charge only liquidity takers. Any liquidity maker then chooses the exchange with the lowest take fee, because it offers the highest quality at the same zero make fee. Our model also allows us to evaluate the consequence of charging liquidity makers only. In this case, competing exchanges are of the same quality and the liquidity maker chooses the exchange with the lowest price (make fee). Charging only one side changes the two-sided price competition to one-sided price competition, which leads competing operators to undercut each other toward zero make and take fees. Also, no operators have incentives to establish multiple exchanges, because the liquidity maker always chooses the exchange with the lowest fee.

The proposal to split the total fee equally between a liquidity maker and a liquidity taker also changes two-sided price competition to one-sided price competition. When such a restriction is imposed, a high-quality (low take fee) exchange must also charge a low price (or low make fee), which destroys the exchange's ability to balance price and quality with respect to the liquidity maker. The liquidity maker thus chooses the exchange with the lowest total fee, which in turn results in competing operators undercutting each other toward a zero total fee, and no operators having an incentive to establish multiple exchanges.

Proposition 7 provides a possible rationale for the differing stances of operators on the policy debate over fee schedules. A major concern of traditional stock exchanges such as the NYSE is the loss of market share to new market entrants such as BATS. Under two-sided price competition, the proof of Proposition 3 demonstrates that no operator can prevent its rivals from making strictly positive profits without losing money itself. One-dimensional price

[31] See Matt Lyons, Senior V.P. & Global Trading Manager of the Capital Group, Panel Discussion at SEC on October 27, 2015, available at: https://www.sec.gov/news/otherwebcasts/2015/equity-market-structure-advisory-committee-102715.shtml.

[32] Ratterman, fn 9.

[33] When $N = 1$, charging one side only or splitting the total fee equally results in no trade (Lemma 1). Thus, all exchanges make zero profit. The proposition holds trivially.

competition, however, can drive profits toward zero. This provides a plausible explanation of why BATS aggressively opposes one-dimensional price competition, as its major source of revenue comes from make/take fees. The NYSE, however, obtains revenue from stock listings as well as fees; this additional revenue could help it to survive short-run zero profit from make/take fees. To be sure, the opposing positions taken by the NYSE and BATS in the policy debate can be driven by other considerations, but the extant literature has yet to explain their divergence of opinion.[34]

Surprisingly, having competition to drive exchange profit to zero may not necessarily improve social welfare under a discrete tick size. As demonstrated in Lemma 1, suppose that the tick size equals 1 and the liquidity maker's valuation is $v_b \in [\frac{1}{2}, 1]$ while the liquidity taker's valuation is $v_s \in [0, \frac{1}{2}]$. Charging one side while subsidizing the other creates a new price level within a tick that can facilitate trades, while charging one side or equal splitting results in no trades occurring, and consequently a loss of social welfare. By contrast, suppose that the tick size equals $\frac{1}{2}$, and the liquidity maker's and liquidity taker's valuations stay the same as in the previous case but they are now separated by the price grid $P = \frac{1}{2}$. In this case, charging no fees to either side facilitates all trading on the price grid $\frac{1}{2}$, which improves social welfare. In the real world, the liquidity maker's and liquidity taker's valuations can either fall within a tick or be separated by a price grid. Under a discrete tick size, the overall effect of having zero make and take fees on social welfare is mixed.

### *IX.B. Implications for the Proliferation of Order Types and Dark pools*

The nature of the fee game is that the operator proposes sub-tick price increments within a given tick. Such an interpretation also sheds light on the proliferation of new order types and dark pools. SEC Rule 612(c) imposes the tick size on displayed orders, but operators can create sub-penny prices using hidden order types or by creating trading systems in which the quotes are hidden. For example, midpoint peg orders have a nominal price at the midpoint, while the midpoint dark pool matches buyers and sellers using the midpoint of the national best bid and offer prices. The pricing of these two mechanisms is similar to a fee structure of

[34] For example, the NYSE argues that brokers tend to route orders to exchanges with the highest rebates, because brokers are not required to pass on the rebates to customers. Yet it is puzzling why the NYSE does not simply match the rebates offered by BATS as a competing device, and instead calls on regulators to ban the rebates. BATS argues that the NYSE's proposal to ban the rebate is anticompetitive, but it is not clear why price competition in a one-dimensional configuration is less competitive than in a two-dimensional configuration.

($f_m = 0.5, f_t = -0.5$) in our model. This fee structure facilitates trading between any liquidity maker and any liquidity taker with valuations specified in our model. In a competitive environment, however, the fee structure ($f_m = 0.5, f_t = -0.5$) cannot preempt the entry of other fee structures, other order types, or the dark pools implied by the fee structure, as is posited in Proposition 3.

More importantly, new order types and dark pools are run by for-profit institutions that charge service fees, so the implied fee structures are different from ($f_m = 0.5, f_t = -0.5$). We are aware that new order types and dark pools can serve other purposes, such as hiding information. Nevertheless, our model provides one explanation for the proliferation of new order types and dark pools. This intuition is consistent with the empirical evidence reported by Kwan, Masulis, and McInish (2015), that the market share of a dark pool increases with the relative tick size (a one-cent tick size divided by the stock price). Also, we provide one plausible explanation why new order types and dark pools do not drive public exchanges and regular limit and market orders out of business, despite their ability to provide transaction costs within the tick.

## X. CONCLUSION

In this paper, we examine make/take fee competition between stock exchanges. When traders can quote a continuous price, the breakdown of the make/take fees is neutralized and order flow consolidates to the exchange with the lowest total fee. Under tick size constraints, fee breakdowns are no longer neutral, and two-sided competition over make and take fees leads to the proliferation of stock exchanges offering almost identical services but charging dispersed fees. We first show that the two-sidedness of such a market allows operators to establish multiple, intrinsically identical, exchanges with heterogeneous fee structures for the purpose of second-degree price discrimination. Second, we demonstrate the non-existence of pure-strategy equilibrium in the fee game under tick size constraints, which leads to fee diversity and frequent fluctuations. Mixed-strategy equilibrium entails positive profits for all competing operators, which encourages the entry of exchanges with new fee structures.

This understanding of the nature of make/take fee competition helps in the evaluation of a recent proposal to ban maker/taker pricing model. There are three arguments for banning the make/take fee pricing model: fairness, complexity, and agency issues. The fairness argument claims that fees lead to wealth transfer from one side of the market to the other. We show, however, that in the presence of tick size constraints the fees imposed by the exchanges can improve welfare by creating sub-tick transaction prices. The second argument for banning the fee cites its complexity and frequent fluctuations. Since the literature provides limited economic rationale for this complexity, Dolgopolov (2014) and Lewis (2014) conjecture that such complexity may serve fraudulent practices to disproportionately benefit certain market participants (SEC 2015). However, the mixed-strategy equilibrium in the model provides an alternative interpretation of this complexity. The final argument for banning the fee involves agency concerns. Battalio, Corwin, and Jennings (2015) find that broker/dealers have a strong incentive to route customers' limit orders to the market offering the highest rebate, because brokers/dealers are permitted to keep such rebates. This conflict of interest leads to two policy proposals: (i) passing the rebate back to customers; and (ii) eliminating the fees (Angel, Harris, and Spatt 2010, 2013). This paper reveals economic forces that favor the first solution. Passing the rebate back to customers is a direct solution to the agency issue, while eliminating the fees might hinder the would-be efficiency of trading within the tick size. Finally, our model predicts that charging one side or splitting fees equally would drive make and take fees toward zero, but the welfare effects of these proposals are mixed under tick size constraints.

We also show that make/take fees are responses from competing exchanges seeking to bypass the *existing* tick size regulations.[35] This result questions the rationale of a recent initiative to increase the tick size for small stocks to five cents. Encouraged by the 2012 JOBS Act, the SEC plans to implement a pilot program to increase the tick size on May 6, 2016.[36] The motivation for increasing the tick size is that it may increase market-making profits, support sell-side equity research, and, eventually, increase the number of IPOs (Weild, Kim, and Newport 2012). We show, however, that exchanges can use fee structures to create cum fee prices that fall within the tick. An increase in the tick size will potentially create more room for multiple exchanges to co-exist and lead to a more fragmented market.

Our model is based on the stock exchange industry, but it reveals two economic mechanisms that can potentially be extended to other industries. First, non-neutrality in our paper stems from the tick size regulation in the exchange industry, but we believe this economic insight applies to other market frictions that also generate non-neutrality. For example, non-surcharge provisions in credit card contracts discourage merchants from charging differentiated prices for payments using different credit cards. The similarity between credit cards and

[35] For studies on the tick size, see Kandel and Marx (1997), Kalay and Anshuman (1998), Buti et al. (2014), O'Hara, Saar, and Zhong (2015), and Yao and Ye (2015), among others.

[36] See SEC press release "SEC Approves Pilot to Assess Tick Size Impact for Smaller Companies" on May 6, 2015, available at http://www.sec.gov/news/pressrelease/2015-82.html.

make/take fees lies in market designs that prevent the two user groups from neutralizing the breakdown of the fees. It would be interesting to explore whether gradual removal of non-surcharge provision would reduce the variety of credit cards.[37] For example, cards with high rebates to customers generally have lower acceptance rates because they usually charge a high fee to merchants. Suppose that merchants can impose surcharges on transactions to recover the fees they are charged. Credit card issuers would compete over the sum of charges to both merchants and card-holders, which could reduce the variety of credit cards. Second, the non-existence of a pure-strategy equilibrium in our model is driven by the simultaneous choice of price and quality. In our paper, quality refers to another price (execution probability implied by the cum fee price for the liquidity taker), but the predictability of the model should hold if quality is a product feature that can be easily adjusted. Such a feature may be hard to find in manufacturing, but we believe it may exist in financial or service-oriented industries.

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[37] Since January 27, 2013, 40 U.S. states have allowed merchants to add a surcharge to purchases paid with a Visa or MasterCard. However, the competition between credit cards is still two sided. First, this settlement does not cover all states. Second, this settlement does not cover all cards. A merchant that accepts American Express cannot surcharge Visa and MasterCard credit cards. Most importantly, merchants are not allowed to discriminate or surcharge based on the involved payment card issuer (i.e. national bank, state bank, credit union, etc.), or whether the card is a rewards card or a non-rewards card.

# APPENDIX A: PROOFS

**Proof of PROPOSITION 1**

Part (i) follows from Equation (2), which implies that for any $v_b \geq T^i$, the maximal surplus for a buyer with valuation $v_b$ to submit a limit order to exchange (hereafter, EX) *i* is

$$BS^{i*}(v_b) = \left(v_b - P^{i*}(v_b) - f_m^i\right) \cdot \text{Prob}\left(v_s \leq min\left\{P^{i*}(v_b) - f_t^i, \frac{1}{2}\right\}\right)$$

$$= \left(v_b - T^i\right) \cdot min\left\{\frac{v_b - T^i}{2}, \frac{1}{2}\right\}$$

$$= \frac{\left(v_b - T^i\right)^2}{2} \text{ (because } v_b - T^i \leq 1 \text{ for any } T^i \geq 0\text{),}$$

which depends only on total fee $T^i$. Consequently, the liquidity maker chooses the EX with the lowest total fee. Since the liquidity taker follows the liquidity maker's choice of EX, she also chooses the EX with the lowest total fee.

For any given $v_b$, the EX with the lowest total fee earns a profit

$$\pi\left(T^i, v_b\right) = \begin{cases} T^i \cdot \text{Prob}\left(v_s \leq \min\left\{P^{i*}(v_b) - f_t^i, \frac{1}{2}\right\}\right) & if\ v_b \geq T^i \\ 0 & if\ v_b < T^i \end{cases}.$$

Accordingly, its total expected profit is given by

$$\pi^{cont} = \int_{v_b \geq T^i} 2 \cdot \pi\left(T^i, v_b\right) dv_b$$

$$= 2T^i \cdot \int_{\max\left\{T^i,\ \frac{1}{2}\right\}}^{1} \text{Prob}\left(v_s \leq \min\left\{P^{i*}(v_b) - f_t^i, \frac{1}{2}\right\}\right) dv_b$$

$$= 2T^i \cdot \int_{\max\left\{T^i,\ \frac{1}{2}\right\}}^{1} \text{Prob}\left(v_s \leq \frac{v_b - T^i}{2}\right) dv_b$$

$$= 2T^i \cdot \int_{\max\left\{T^i,\ \frac{1}{2}\right\}}^{1} (v_b - T^i) dv_b$$

which is a function of $T^i$, and does not depend on how $T^i$ is decomposed into $f_m^i$ and $f_t^i$. EXs whose total fees are not the lowest earn zero profits.

Thus for all three types of players, only the total fee matters.

Parts (ii) and (iii) follow directly from part (i). Since all traders go to the EX with the lowest total fee, no operator can increase its profit by establishing multiple exchanges with heterogeneous total fee. When multiple operators compete, they undercut each other to zero total fee and earn zero profits.

To solve for optimal monopoly pricing, note that a monopoly operator does not have any incentive to open multiple EXs, because any liquidity maker chooses the EX with the lowest total fee. So using the formula for $\pi^{cont}$ above (we remove superscript $i$ to emphasize it is monopoly here), the monopoly EX's profit can be written as

$$\pi^{contM}(T) = 2T \cdot \int_{\max\{T,\ \frac{1}{2}\}}^{1} (v_b - T) dv_b = \begin{cases} T \cdot (\frac{3}{4} - T) & if \ \ \frac{1}{2} \geq T \\ T \cdot (1-T)^2 & if \ \ \frac{1}{2} < T \end{cases}$$

The solutions are $T^{contM*} = \frac{3}{8}$ and $\pi^{contM*} = \frac{9}{64}$. ■

**Proof of LEMMA 1**

Part (i) follows from the necessary condition identified by Equation (5), for a trade to occur. If the make fee $f_m > 0$, the liquidity maker must propose a limit-order price at $P \leq 0$, because otherwise $P + f_m > 1$. It then follows that for a trade to happen, the take fee $f_t < 0$, because otherwise $P - f_t \leq 0$.

Similarly, if the make fee $f_m < 0$, the liquidity maker must propose a limit-order price at $P \geq 1$ and the take fee $f_t > 0$. In summary, a necessary condition for a trade to occur is charging one side and subsidizing the other.

In addition, Equation (5) implies that, for a trade to occur, we need $\begin{cases} 0 < P - f_t \\ P + f_m < 1 \end{cases}$, which is equivalent to $\begin{cases} f_t < P \\ f_m < 1 - P \end{cases}$. Thus, $f_m + f_t < 1$. For the EX to survive, we need $f_m + f_t \geq 0$.

Part (ii) follows from part (i) and $|f_i| \leq 1$ ($i$=$m$,$t$). In particular, if $f_m > 0$, from the proof in part (i), we have $P \leq 0$ and $f_t < 0$. Equation (5) and $|f_i| \leq 1$ ($i$=$m$,$t$) together yield $-1 \leq f_t < P$. So $P = 0$. In parallel, we can show $P = 1$ if $f_m < 0$.

Part (iii) follows directly from the definitions of the cum fee buy and sell prices. ■

**Proof of LEMMA 2**

Under EX 1's fee structure $(\alpha, \beta)$, according to parts (ii) and (iii) of Lemma 1, the liquidity maker proposes a buy limit order at price $P = 0$, and trade with the liquidity taker with valuation $v_s \leq \min\{-\beta, \frac{1}{2}\}$. So the liquidity maker's surplus when choosing EX 1 is

$$BS^1 = (v_b - \alpha) \cdot \text{Prob}\left(v_s \leq \min\left\{-\beta, \frac{1}{2}\right\}\right)$$

$$= \begin{cases} (v_b - \alpha) \cdot 2(-\beta) & \text{if } -\beta < \frac{1}{2} \\ v_b - \alpha & \text{if } -\beta \geq \frac{1}{2} \end{cases}.$$

Similarly, under EX 2's fee structure $(\beta, \alpha)$, the liquidity maker proposes a buy limit order at price $P = 1$, and trade with the liquidity taker with valuation $v_s \leq \min\{1-\alpha, \frac{1}{2}\}$. So the liquidity maker's surplus when choosing EX 2 is

$$BS^2 = (v_b - 1 - \beta) \cdot \text{Prob}(v_s \leq \min\{1-\alpha, \frac{1}{2}\})$$

$$= \begin{cases} (v_b - 1 - \beta) \cdot 2(1-\alpha) & \text{if } \alpha > \frac{1}{2} \\ v_b - 1 - \beta & \text{if } \alpha \leq \frac{1}{2} \end{cases}.$$

We consider the following three possible cases:

Case (i): $\alpha > -\beta \geq \frac{1}{2}$

$$\Delta BS(v_b) \equiv BS^1 - BS^2 = v_b - \alpha - (v_b - 1 - \beta) \cdot 2(1-\alpha)$$

$$= 2 \cdot \left[\left(\alpha - \frac{1}{2}\right) \cdot v_b - \frac{1}{2} \cdot \alpha + (1+\beta) \cdot (1-\alpha)\right].$$

Note that $\Delta BS(v_b)$ increases with $v_b$, because $\alpha > \frac{1}{2}$. Hence,

$$\Delta BS(v_b) \leq \Delta BS(1)$$

$$= 2 \cdot (1-\alpha) \cdot (\frac{1}{2} + \beta) \leq 0.$$

The liquidity maker thus prefers EX 2.

Case (ii): $\alpha > \frac{1}{2} > -\beta$

$$\Delta BS(v_b) = 2 \cdot (v_b - \alpha) \cdot (-\beta) - (v_b - 1 - \beta) \cdot 2(1-\alpha)$$

$$= 2 \cdot (\alpha - \beta - 1) \cdot (v_b - 1).$$

So

$$\Delta BS(v_b) \gtreqless 0 \text{ if and only if } \alpha - \beta \lesseqgtr 1.$$

Case (iii): $\frac{1}{2} \geq \alpha > -\beta$

$$\Delta BS(v_b) = 2 \cdot (v_b - \alpha) \cdot (-\beta) - v_b - 1 - \beta$$

$$= 2 \cdot \left[\left(-\beta - \frac{1}{2}\right) \cdot v_b - (-\beta) \cdot \alpha + (1+\beta) \cdot \frac{1}{2}\right].$$

Note that $\Delta BS(v_b)$ decreases with $v_b$, because $\frac{1}{2} > -\beta$. Hence,

$$\Delta BS(v_b) \geq \Delta BS(1)$$

$$= 2 \cdot (-\beta) \cdot (\frac{1}{2} - \alpha) \geq 0.$$

The liquidity maker thus prefers EX 1.

Combining cases (i)–(iii), Lemma 2 follows. ■

**Proof of PROPOSITION 2**

Suppose a monopoly operator opens $k$ EXs. Denote $p_b^i$ and $p_s^i$ as EX $i$'s cum fee buy price and cum fee sell price, respectively, where $1 \leq i \leq k$. Without loss of generality, we can assume that $p_s^1 \leq p_s^2 \leq \cdots \leq p_s^k$. The analysis of product differentiation in Section V suggests that $p_b^1 \leq p_b^2 \leq \cdots \leq p_b^k$. If not, suppose that EX $i$ has a lower cum fee buy price and a higher cum fee sell price than EX $j$. Then no liquidity maker chooses EX $j$. Moreover, the cum fee sell price $p_s^i \leq \frac{1}{2}$ for all $i$, because whenever the cum fee sell price $p_s^i > \frac{1}{2}$, EX $i$ can increase its profit by lowering $p_s^i$ by a small amount, so that it earns a higher per-transaction fee $p_b^i - p_s^i$ but still has the full participation of liquidity takers.

Now we analyze liquidity makers' segmentation across these $k$ EXs. A liquidity maker with valuation $v_b < p_b^1$ posts no limit order on any EX. A liquidity maker with valuation $p_b^1 \leq v_b < p_b^2$ posts a limit order on EX 1. A liquidity maker with valuation $p_b^2 \leq v_b < p_b^3$ posts a limit order on EX 1 only if

(A.1) $$2 \cdot (v_b - p_b^2) \cdot p_s^2 \leq 2 \cdot (v_b - p_b^1) \cdot p_s^1,$$

which is equivalent to $v_b \leq \frac{p_b^2 p_s^2 - p_b^1 p_s^1}{p_s^2 - p_s^1}$. Denote $\varphi^i = \frac{p_b^2 p_s^2 - p_b^1 p_s^1}{p_s^2 - p_s^1}$. It follows that a liquidity maker with valuation $v_b \epsilon [p_b^1, \varphi^1]$ posts a limit order on EX 1. Similarly, denote

$$\varphi^i = \begin{cases} p_b^1 & \text{for } i = 0 \\ \dfrac{p_b^{i+1} p_s^{i+1} - p_b^i p_s^i}{p_s^{i+1} - p_s^i} & \text{for } 1 \leq i \leq k-1 \\ 1 & \text{for } i = k \end{cases}$$

A liquidity maker with valuation $v_b \epsilon \left[\varphi^i, \varphi^{i-1}\right]$ posts a limit order on EX $i$ for $1 \leq i \leq k$.

The monopoly operator chooses a cum fee buy price $p_b^i$ and cum fee sell price $p_s^i$ on each EX *i* to maximize

$$\pi^M(k) = 4 \sum_{i=1}^{k} (\varphi^i - \varphi^{i-1}) \cdot p_s^i \cdot (p_b^i - p_s^i)$$

$$\text{s.t. } \varphi^0 \leq \varphi^1 \leq \cdots \leq \varphi^k$$

We first maximize the profit function above *without* constraint $\varphi^0 \le \varphi^1 \le \cdots \le \varphi^k$, which yields the optimal cum fee buy and sell prices for each EX $i$, given by Equation (14).

Under the cum fee buy and sell prices from Equation (14), we have

$$\varphi^i = \frac{k+i+1}{2k+1},\ i = 0, 1, \cdots, k$$

which satisfies constraint $\varphi^0 \le \varphi^1 \le \cdots \le \varphi^k$. So the cum fee buy and sell prices given by Equation (14) are also solutions to the constrained maximization problem. By establishing $k$ EXs, the profit of the monopoly operator is

$$\pi^M(k) = 4\sum_{i=1}^{k}\left(\frac{\mathrm{k+i+1}}{\mathrm{2k+1}} - \frac{k+i}{2k+1}\right)\cdot\frac{i}{2k+1}\cdot\left(\frac{1}{2}+\frac{i}{2(2k+1)}-\frac{i}{2k+1}\right)$$

$$= \frac{2k\cdot(k+1)}{3(2k+1)^2}$$

The liquidity maker's expected surplus is

$$BS^M(k) = 4\sum_{i=1}^{k}(\varphi^i - \varphi^{i-1})\cdot p_s^i\cdot\left(E\left(v_b\middle|\varphi^{i-1}\le v_b\le\varphi^i\right)-p_b^i\right) = \frac{k\cdot(k+1)}{3(2k+1)^2}$$

The liquidity taker's surplus is

$$SS^M(k) = 4\sum_{i=1}^{k}(\varphi^i - \varphi^{i-1})\cdot p_s^i\cdot(p_s^i - E\left(v_s\middle|v_s\le p_s^i\right)) = \frac{k\cdot(k+1)}{3(2k+1)^2}$$

If a fixed cost $c$ is involved in opening a new EX, the monopoly operator keeps establishing the $k^{th}$ EX as long as

$$\pi^M(k) - \pi^M(k-1) = \frac{4k}{3(2k+1)^2(2k-1)^2} \ge c.\ \blacksquare$$

**Proof of PROPOSITION 3**

We prove this proposition through contradiction. Without loss of generality, suppose that pure-strategy equilibrium exists and that $\pi^1 \ge \pi^2$ in equilibrium. We prove that a profitable deviating strategy always exists for EX 2, so the pure strategy cannot be sustained in equilibrium. Two cases are to be considered: (i) $\pi^1 > 0$; (ii) $\pi^1 = \pi^2 = 0$.

(i) There are two subcases to be considered: (i-a) $\pi^1 > \pi^2 \ge 0$; (i-b) $\pi^1 = \pi^2 > 0$.

(i-a) EX 2 can set its fees such that

(A.2) $$p_b^2 = p_b^1 - \varepsilon \text{ and } p_s^2 = p_s^1,$$

where $\varepsilon > 0$. Then any liquidity maker chooses EX 2, and EX 2's profit becomes

$$\begin{aligned}\hat{\pi}^2 &= (p_b^1 - \varepsilon - p_s^1) \cdot \text{Prob}(v_b \geq \hat{v}_b^2) \cdot \text{Prob}(v_s \leq \hat{v}_s^2) \\ &\geq (p_b^1 - p_s^1 - \varepsilon) \cdot \text{Prob}(v_b \geq \hat{v}_b^1) \cdot \text{Prob}(v_s \leq \hat{v}_s^1) \\ &= \pi^1 - \varepsilon \cdot \text{Prob}(v_b \geq \hat{v}_b^1) \cdot \text{Prob}(v_s \leq \hat{v}_s^1),\end{aligned}$$

where the inequality follows from (A.2), $\hat{v}_s^i$ is given by Equation (9), and $\hat{v}_b^i \equiv \max\{p_b^1, \frac{1}{2}\}$. Clearly, as long as $\pi^1 > \pi^2$, EX 2 can always strictly increase its profit by using the deviating strategy (A.2) with a sufficiently small $\varepsilon$.

(i-b) If EX 2 and EX 1 have the same cum fee sell price, they must have the same cum fee buy price to obtain equal profit. It is then easy to see that deviation (A.2) nearly doubles EX 2's profit. Now consider the case that they have heterogeneous cum fee sell prices but the same profit. Without loss of generality, suppose that EX 2 starts with higher quality than EX 1. Then EX 2 must have higher cum fee buy price than EX 1 does. Liquidity makers with relatively higher valuations choose EX 2 while liquidity makers with relatively lower valuation choose EX 1.

Deviation (A.2) of EX 2 first attracts low-valuation liquidity makers with infinitesimal profit concession. Deviation (A.2) also retains EX 2's original liquidity makers. They choose to participate because the new cum fee buy price is lower than $p_b^1$, which is lower than EX 2's original cum fee buy price and (A.2) is better than $(p_b^1, p_s^1)$. EX 2 can choose $\varepsilon$ sufficiently small such that the expected profit from retaining high-valuation liquidity makers is greater than the infinitesimal decrease in profit from low-valuation liquidity makers.

Therefore, deviation (A.2) rules out the possibility that two EXs earn the same positive profits in equilibrium.

(ii) There are two subcases that two EXs earn zero profits: (ii-a) no trading; (ii-b) trading with $p_b^i = p_s^i \in (0,1), i = 1,2$.

(ii-a) No trading implies that $p_b^i \geq 1$ or $p_s^i \leq 0$ for $i = 1,2$. Then EX 2 earn positive profit by setting $0 < p_s^2 < p_b^2 < 1$ such that liquidity makers with valuation $v_b \geq \hat{v}_b^2$ trade with the liquidity taker with valuation $v_s \leq \hat{v}_s^2$.

(ii-b) Denote $p_b^1 = p_s^1 = a$. Two further subcases are to be considered: (ii-b-I) $0 < a < \frac{1}{2}$; (ii-b-II) $\frac{1}{2} \leq a < 1$.

(ii-b-I) EX 2 can set its fees such that

(A.3) $$p_b^2 = a + \varepsilon \text{ and } p_s^2 = a + \mu \cdot \varepsilon,$$

where $\varepsilon > 0$ and $0 < \mu < 1$. For sufficiently small $\varepsilon$, we have $\hat{v}_s^2 = p_s^2 > a = \hat{v}_s^1$ and $p_b^2 > p_b^1$. From Equation (13) in Section V.B., the liquidity maker's segmentation cutoff is

$$\varphi = \frac{p_b^2 \cdot p_s^2 - p_b^1 \cdot p_s^1}{p_s^2 - p_s^1}.$$
$$= a\left(1 + \frac{1}{\mu}\right) + \varepsilon.$$

Note that $\varphi$ decreases with $\mu$. For $\mu$ sufficiently close to 1 and $\varepsilon$ sufficiently close to zero, $\lim_{\mu \to 1} \varphi = 2a + \varepsilon < 1$, where the inequality follows from $a < \frac{1}{2}$. Based on the liquidity makers' segmentation in Section V.B., a liquidity maker with valuation $v_b \in (\varphi, 1]$ chooses EX 2, and trades with a liquidity taker with valuation $v_s \in (0, a + \mu \cdot \varepsilon]$. EX 2 earns a strictly positive profit $p_b^2 - p_s^2 = (1 - \mu) \cdot \varepsilon$ from such liquidity makers and liquidity takers.

(ii-b-II) EX 2 can set its fees such that

(A.4) $$p_b^2 = a - \mu \cdot \varepsilon \text{ and } p_s^2 = a - \varepsilon,$$

where $\varepsilon > 0$ and $0 < \mu < 1$.

When $a > \frac{1}{2}$, for sufficiently small $\varepsilon$, we can always have $p_s^2 = a - \varepsilon \geq \frac{1}{2}$ and $p_b^2 < p_b^1$. This implies that any liquidity maker chooses EX 2, and EX 2 earns a strictly positive profit, as $p_b^2 - p_s^2 = (1 - \mu) \cdot \varepsilon > 0$.

When $a = \frac{1}{2}$, for sufficiently small $\varepsilon$, we have $\hat{v}_s^2 = p_s^2 < a = \hat{v}_s^1$ and $p_b^2 < p_b^1$. The liquidity maker's segmentation cutoff is

$$\varphi = \frac{p_b^2 \cdot p_s^2 - p_b^1 \cdot p_s^1}{p_s^2 - p_s^1}.$$
$$= \frac{1}{2} + \mu \cdot \left(\frac{1}{2} - \varepsilon\right).$$

For sufficiently small $\varepsilon$, $\varphi > \frac{1}{2}$. From the liquidity makers' segmentation analysis in Section V.B., liquidity makers with valuation $\frac{1}{2} \leq v_b \leq \varphi$ choose EX 2. EX 2 earns a strictly positive profit, as $p_b^2 - p_s^2 = (1 - \mu) \cdot \varepsilon > 0$. ■

**Proof of PROPOSITION 4**

Part (i): We establish the existence of symmetric mixed-strategy equilibrium by applying Theorem 6* in Dasgupta and Maskin (1986) (hereafter, D-M).

***Theorem 6* (D-M, p. 24).*** *Let $\bar{\bar{A}} \subseteq R^m$ be non-empty, convex and compact, and let $[(\bar{\bar{A}}, U_i); i = 1, \ldots, N]$ be a symmetric game, where $\forall i, U_i: \underbrace{\bar{\bar{A}} \times \ldots \times \bar{\bar{A}}}_{N\ times} \to R^1$ is continuous,* *except on a subset $A^{**}(i)$ of $A^{*}(i)$, where $A^{*}(i)$ is defined by (A1). Suppose $\sum_{i=1}^{N} U_i(a)$ is upper semicontinuous, and for all $i, U_i$ is bounded and satisfies Property ($\alpha^*$).*

*Then there exists a symmetric mixed-strategy equilibrium ($\mu^*, \dots, \mu^*$) with the property that $\forall i$ and $\forall \bar{a}_i \epsilon A_i^{**}(i), \mu^*(\{\bar{a}_i\}) = 0$.*

Condition *(A1)* and *Property ($\alpha^*$)* are as follows in D-M (1986):

***(A1)*** *Let $Q \subseteq \{1, \dots, m\}$. For each pair of agents $i, j \in \{1, \dots, N\}$, let $D(i)$ be a positive integer. For each integer $d$, with $1 \le d \le D(i)$, let $f_{ij}^{d}: R^1 \to R^1$ be a one-to-one, continuous function with the property that $f_{ij}^{d} = (f_{ij}^{d})^{-1}$. Finally, $\forall i$, define*

$$A^*(i) = \left\{(a_1, \dots, a_N) \in A \middle| \exists j \neq i, \exists k \in Q, \exists d, 1 \le d \le D(i), \text{such that } a_{jk} = f_{ij}^{d}(a_{ik})\right\}.$$

***Property ($\alpha^*$).*** $\forall \bar{a}_i \epsilon A_i^{**}(i), \exists$ *a non-atomic measure $\nu$ on $B^m$ such that for all $\boldsymbol{a_{-i}}$ that $(\bar{a}_i, \boldsymbol{a}_{-i}) \epsilon A^{**}(i)$*

$$\oint_{B^m} \left[\liminf_{\theta \to 0} U_i(\bar{a}_i + \theta e, \boldsymbol{a_{-i}})\, d\nu(\mathrm{e})\right] \ge U_i(\bar{a}_i, \boldsymbol{a_{-i}})$$

*where the inequality is strict if*

$$\boldsymbol{a_{-i}} = \underbrace{\bar{a}_i \times \dots \times \bar{a}_i}_{(N\text{-}1)\ times}$$

We now confirm that the game satisfies the conditions of Theorem 6* in D-M.

Our game has two EXs, and each EX sets its cum fee buy and sell price $a_i = (p_b^i, p_s^i), i = 1,2$. The feasible set of fee structures is given by $\bar{\bar{A}} \equiv \{(p_b, p_s) | p_b - p_s \ge 0, 0 \le p_b \le 1\}$, which is non-empty, convex, compact, and a subset of $R^2$. Each EX's profit function $U_i(a_1, a_2)$ is continuous, except when $a_1 = a_2 = \bar{a}$. $A^{**}(i) = \{(\bar{a}, \bar{a}) | \bar{a} = a_1 = a_2, a_1 \in \bar{\bar{A}}\}$. The $f_{ij}^{d}$ in (A1) is the identity function. It is easy to see that $U_i$ is bounded and that the discontinuities of $U_i$, $A^{**}(i)$, are confined to a subset of $A^{*}(i)$.

If the sum of EXs' profits, $U(a_1, a_2) \equiv U_1(a_1, a_2) + U_2(a_1, a_2)$, is continuous, it is upper semi-continuous. Because $U_i(a_1, a_2)$ is continuous except at $a_1 = a_2 = \bar{a}$, we need to prove the continuity only at $U(\bar{a}, \bar{a})$, where $\bar{a} = (p_b, p_s)$. Denote $e(\eta) = (\cos \eta, \sin \eta), \eta \in [0, 2\pi)$. So $\bar{a} + \theta e = (p_b + \theta \cos \eta, p_s + \theta \sin \eta)$. The following three steps show that for any $\varepsilon > 0$, there always exists $\theta$ such that $U(\bar{a} + \theta e, \bar{a}) - U(\bar{a}, \bar{a}) < \varepsilon$ and $U(\bar{a}, \bar{a} + \theta e) - U(\bar{a}, \bar{a}) < \varepsilon$.

(i) Liquidity makers who still choose fee structure $\bar{a}$ do not affect the total profit.

(ii) Each liquidity maker who prefers fee structure $\bar{a} + \theta e$ over $\bar{a}$ reduces the total profit by $2 \cdot (p_b - p_s) \cdot p_s$ , but increases the total profit by $2 \cdot (p_b + \theta \cos\eta - p_s - \theta \sin\eta) \cdot (p_s + \theta \sin\eta) = 2 \cdot (p_b - p_s) \cdot p_s + 2\theta \cdot [p_b \cdot \sin\eta + p_s \cdot (cos\eta - 2sin\eta) + \theta \cdot (cos\eta - sin\eta) \cdot \eta]$. As the measure of liquidity makers is $\frac{1}{2}$, the total profit change is no more than $\theta \cdot [p_b \cdot \sin\eta + p_s \cdot (cos\eta - 2sin\eta) + \theta \cdot (cos\eta - sin\eta) \cdot \eta]$. So there exists a sufficiently small $\theta$ to ensure that the total profit change is smaller than $\frac{\varepsilon}{2}$.

(iii) The new fee structure $\bar{a} + \theta e$ can also increase total participation. As the cum fee buy price changes at most by $\theta$, the change in probability of total participation on the part of liquidity makers is less than $\frac{\theta}{1-0.5} = 2\theta$. This participation change affects the total profit at most by $1 \cdot 2\theta \cdot 1$, because both liquidity takers' participation probability and the total fee are no more than one. So there exists a sufficiently small $\theta$ to ensure that the total profit change is less than $\frac{\varepsilon}{2}$.

Combining the effects of (i)–(iii), we can always find a sufficiently small $\theta$ to ensure that the total profit change is less than $\varepsilon$.

We finally check that Property ($\alpha^*$) holds in our game. Note that Property ($\alpha^*$) implies that, for any sufficiently small $\theta > 0$, one EX (without loss of generality, suppose it is EX 1) can slightly deviate its fee structure from the discontinuous point to a non-zero measure set on $B^2$ (a circle with radius $\theta$) and receive higher profit to the one at the discontinuity point. If, for any sufficiently small $\theta > 0$, we can find an arc $e_o \in B^2$ (a non-zero measure set) such that for all $e \in e_o$, we have

$$U_1(\bar{a} + \theta e, \bar{a}) > U_1(\bar{a}, \bar{a}),$$

then we can always uniformly distribute all $\nu(e)$ to arc $e_o$ and ensure that the LHS of Property($\alpha^*$) is strictly larger than the RHS of Property($\alpha^*$).

Next, we show the existences of $e_0$ for all possible $\bar{a} = (p_b, p_s)$.

Case (i): $p_b - p_s > 0$

We propose $e_0 = \left\{e(\eta) \middle| \frac{1}{2}\pi < \eta < \pi\right\}$, which implies that $\theta \cos\eta < 0$, and $\theta \sin\eta > 0$. That is, EX 1 undercuts both the make fee and the take fee of EX 2. Hence EX 1 attracts any liquidity maker and its profit increases when $\theta$ is sufficiently small.

Case (ii): $p_b - p_s = 0$

The cutoff valuation of the marginal liquidity maker, $\varphi$, is,

$$(\varphi - p_b - \theta cos\,\eta) \cdot (p_s + \theta sin\,\eta) = (\varphi - p_b) \cdot (p_s)$$

$$\therefore \varphi = p_b \cdot (1 + \cot\eta) + \theta\cos\eta.$$

When $p_b \in [0,0.5)$, $e_0 = \left\{ \mathrm{e}(\eta) \middle| arccot\left(\frac{1}{p_b} - 1\right) < \eta < \frac{1}{4}\pi \right\}$ satisfies the following conditions. [38]

a. $\theta\sin\eta > 0$, which implies that EX 1 is of higher quality than EX 2. A liquidity maker with valuation $v > \varphi$ chooses EX 1.

b. $\varphi < 1$, which guarantees the existence of the liquidity maker who has valuation $\varphi < v_b < 1$ mentioned in (a).

c. $\theta\cos\eta - \theta\sin\eta > 0$, which ensures that the total fee is greater than zero.

Combining (a)–(c), we have

$$U_1(\bar{a} + \theta e, \bar{a})|_{e \in e_0} > 0 = U_1(\bar{a}, \bar{a}).$$

When $p_b \in [0.5,1]$, $e_0 = \left\{ \mathrm{e}(\eta) \middle| \frac{5}{4}\pi < \eta < \frac{3}{2}\pi \right\}$ satisfies the following conditions. [39]

a. $\theta\sin\eta < 0$, which implies that EX 1 is of lower quality than EX 2. A liquidity maker with valuation $v < \varphi$ chooses EX 1.

b. $p_b < \varphi$, which guarantees the existence of the liquidity maker who has valuation $p_b < v_b < \varphi$ mentioned in (a).

c. $\theta\cos\eta - \theta\sin\eta > 0$ , which ensures that the total fee is greater than zero.

Combining (a)–(c), we have

$$U_1(\bar{a} + \theta e, \bar{a})|_{e \in e_0} > 0 = U_1(\bar{a}, \bar{a}).$$

In summary, Property ($\alpha^*$) holds for any $\bar{a}_1 = \bar{a}_2 = (p_b, p_s)$.

Part (ii): It is easy to see that mixed-strategy equilibrium should not entail negative profit. We then prove the non-existence of mixed-strategy equilibrium with exchanges earning zero profit by contradiction.

1. If mixed-strategy equilibrium involves zero profit, the probability that the effective buy price is greater than 0.5 must be zero, that is, $\mathrm{Prob}(x < p_b < y) = 0$ for any $x > 0.5$. Here $p_s \le p_b$.

   If not, one EX can set $\tilde{p}_b = \frac{0.5+x}{2}$ and $\tilde{p}_s = 0.5$. This deviation attracts at least liquidity makers with valuation $v_b \epsilon [\frac{0.5+x}{2}, x]$ and makes positive profits.

[38] This deviation corresponds to $\widetilde{p_b} = p_b + \varepsilon$ , $\widetilde{p_s} = p_s + \mu\varepsilon$ $\left(0 < \varepsilon \text{ and } \frac{p_b}{1-p_b} < \mu < 1\right)$.

[39] This deviation corresponds to $\widetilde{p_b} = p_b - \mu\varepsilon$ , $\widetilde{p_s} = p_s - \varepsilon$ $(0 < \varepsilon$ and $0 < \mu < 1)$.

2. If mixed-strategy equilibrium involves zero profit, the probability that the effective buy price is smaller than 0.5 must be zero, that is, $\text{Prob}(x < p_b < y) = 0$ for any $y < 0.5$.

   If not, one EX can have a profitable deviation by setting its fee structure as $\tilde{p}_b = 0.5$ , $\tilde{p}_s = 0.5 - \varepsilon$ $(\varepsilon > 0)$, because $(\tilde{p}_b, \tilde{p}_s)$ could attract some traders for any $p_b = p_s = c$, where $c \in (x, y)$. The proof is equivalent to showing that, for sufficiently small $\varepsilon$, we can find $\varphi < 1$ such that

$$(\varphi - 0.5)(0.5 - \varepsilon) = (\varphi - c)c$$

$$\varphi = \frac{0.25 - 0.5\varepsilon - c^2}{0.5 - \varepsilon - c}$$

$$\varphi < 1 \Leftrightarrow \varepsilon < 2\,(0.5 - c)^2$$

   Thus we can always find $\varepsilon > 0$ such that $\tilde{p}_b = 0.5$ , $\tilde{p}_s = 0.5 - \varepsilon$ attracts any liquidity maker with valuation $v_b \epsilon [\varphi, 1]$ for any $p_b = p_s = c \in (x, y)$. It follows that such deviation can also outperform any $p_b = c > p_s$, because $p_b = p_s = c$ dominates any $p_b = c > p_s$ for the liquidity maker. Consequently, such a deviation can always yield positive profit with positive probability.
3. If mixed-strategy equilibrium involves zero profit, the probability that the effective buy price is equal to 0.5 must be zero. If not, one EX can use strategy (A.4) to ensure itself positive profit with positive probability.

To summarize, a mixed strategy with zero profit implies that $p_b$ has zero probability to be at any interval of [0, 1], which is a contradiction. ■

**Proof of PROPOSITION 5**

*i) Conditions for the Existence of Proposed Price on Grid:*

Given $v_b$, when the liquidity maker can only quote on the price grids given by *tick size constraints (1')*, her expected surplus is

$$BS(P; v_b) \equiv (v_b - P - f_m) \cdot \text{Prob}(v_s \leq P - f_t)$$

$$= (v_b - P - f_m) \cdot 2 \cdot \max\left\{0, \min\left\{\frac{1}{2}, P - f_t\right\}\right\},$$

where $P = \frac{n}{N}$, and $n$ and $N$ are integers.

To ensure $BS(P; v_b) \geq 0$ so that the liquidity maker will propose a price, we need the existence of some feasible $P = \frac{n}{N}$ to ensure $\begin{cases} 0 \leq v_b - P - f_m \\ 0 \leq P - f_t \end{cases}$. That is, we need some $n \in Z$ s.t. $\begin{cases} \frac{n}{N} \leq v_b - f_m \\ 0 \leq \frac{n}{N} - f_t \end{cases}$, which can be reduced to

(A.5) $$\lceil N \cdot f_t \rceil \leq n \leq \lfloor N \cdot (v_b - f_m) \rfloor$$

For (A.5) to hold, we need $\lceil N \cdot f_t \rceil \leq \lfloor N \cdot (v_b - f_m) \rfloor$, which is equivalent to

(A.6) $$v_b \geq f_m + \frac{\lceil N \cdot f_t \rceil}{N} \ (\because \lceil y \rceil \leq \lfloor x \rfloor \Leftrightarrow \lceil y \rceil \leq x)$$

Since

$\frac{1}{2} \leq v_b \leq 1$, for non-zero participation from liquidity makers, we need

(A.7) $$f_m + \frac{\lceil N \cdot f_t \rceil}{N} < 1.$$

*ii) Optimal Proposed Limit-Order Price $\frac{n(v_b, f_m, f_t)}{N}$:*

When $v_b < f_m + \frac{\lceil N \cdot f_t \rceil}{N}$, the liquidity maker does not submit limit order. When $v_b \geq f_m + \frac{\lceil N \cdot f_t \rceil}{N}$, the liquidity maker chooses a grid to maximize her expected surplus $BS(P; v_b) = (v_b - P - f_m) \cdot 2 \cdot \max\left\{0, \min\left\{\frac{1}{2}, P - f_t\right\}\right\}$

Note that $BS(P; v_b)$ is essentially a quadratic function of $P$, as shown in Figure A1. When $P$ can be any real number, the optimal price is $\frac{v_b - f_m + f_t}{2}$.

Due to tick size constraints (1'), $P$ can only be at price grids $\frac{n}{N}$. Hence, when (A.6) holds, a liquidity maker with valuation $v_b$ proposes a limit order at a price grid $\frac{n}{N}$ that is closest to the unconstrained optimal price $\frac{v_b - f_m + f_t}{2}$. So $P^*(v_b) \equiv \frac{n}{N}$ if and only if $\frac{n - \frac{1}{2}}{N} \leq \frac{v_b - f_m + f_t}{2} < \frac{n + \frac{1}{2}}{N}$,

which is equivalent to

$$\frac{N \cdot (v_b - f_m + f_t)}{2} - \frac{1}{2} < n \leq \frac{N \cdot (v_b - f_m + f_t)}{2} + \frac{1}{2}.$$

Thus

$$n(v_b, f_m, f_t) = \left\lfloor \frac{N \cdot (v_b - f_m + f_t) + 1}{2} \right\rfloor.$$

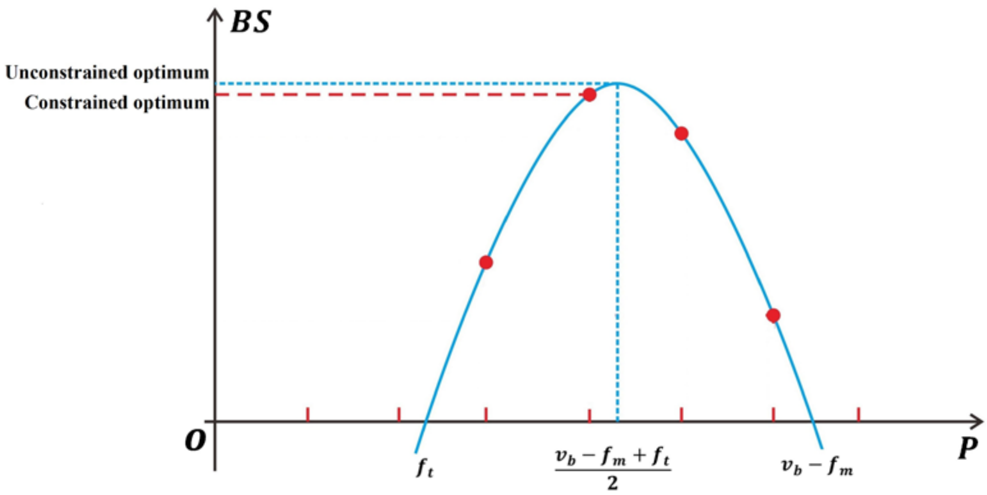


**FIGURE A.1**

**Optimal Limit-Order Price under Multiple Ticks**

*Notes*: This figure illustrates the decision of the liquidity maker with valuation $v_b$. The horizontal axis represents the price level and the vertical axis represents the liquidity maker's expected surplus. Each red dot on the parabola represents a price level in the grid that gives the liquidity maker a positive expected surplus. The optimal price in the grid is the one closest to $\frac{v_b - f_m + f_t}{2}$, which is the liquidity maker's optimal price without tick size constraints.

*iii) Equivalence of fee structure $(f_m, f_t)$ and fee structure $(\tilde{f}_m, \tilde{f}_t)$:*

For any given $(f_m, f_t)$, we have $n(v_b, f_m, f_t) = \left\lfloor \frac{N \cdot (v - f_m + f_t) + 1}{2} \right\rfloor$, and thus the cum fee buy and sell prices are $p_b = \frac{n(v_b, f_m, f_t)}{N} + f_m$, and $p_s = \frac{n(v_b, f_m, f_t)}{N} - f_t$.

Now consider $(\tilde{f}_m, \tilde{f}_t)$, where $\begin{cases} \tilde{f}_m = f_m + \frac{k}{N} \\ \tilde{f}_t = f_t - \frac{k}{N} \end{cases}$, $k \in Z$. Under $(\tilde{f}_m, \tilde{f}_t)$, $n\left(v_b, \tilde{f}_m, \tilde{f}_t\right) = \left\lfloor \frac{N \cdot (v - \tilde{f}_m + \tilde{f}_t) + 1}{2} \right\rfloor = \left\lfloor \frac{N \cdot (v - f_m + f_t) + 1}{2} - k \right\rfloor = n(v_b, f_m, f_t) - k$. It follows that $\tilde{p}_b = \frac{n(v_b, \tilde{f}_m, \tilde{f}_t)}{N} + \tilde{f}_m = \frac{n(v_b, f_m, f_t)}{N} - \frac{k}{N} + f_m + \frac{k}{N} = p_b$, and $\tilde{p}_s = \frac{n(v_b, \tilde{f}_m, \tilde{f}_t)}{N} - \tilde{f}_t = \frac{n(v_b, f_m, f_t)}{N} - \frac{k}{N} - f_t + \frac{k}{N} = p_s$ . Since, for any given $v_b$, the cum fee buy and sell prices are the same under the fee structure $(f_m, f_t)$ and the fee structure $(\tilde{f}_m, \tilde{f}_t)$, these fee structures are equivalent for the liquidity maker, the liquidity taker and EXs. ■

**Proof of PROPOSITION 6**

We prove this proposition through contradiction. It is easy to see that no exchange should take a negative profit in pure-strategy equilibrium, because the operator can simply shut down the exchange. The proof is then divided into two steps.

*Step 1: No exchange should take a positive profit in pure-strategy equilibrium.*

Suppose not: denote the total number of EXs that take positive profits as $H$. As we have more than one operator, we can always find an Operator $i$ who does not own all $H$ exchanges, that is, her profit is lower than the combined profits of these $H$ exchanges.

Operator $i$ can increase her profits by establishing $H$ EXs to undercut each of the existing profitable EXs with the following fee structure:

$$f_m^{ih} = f_m^h - \varepsilon \text{ and } f_t^{ih} = f_t^h, \text{ for } h = 1,2,3,\cdots,H.$$

Such a deviation allows Operator $i$ to capture the entire market. Therefore, step 1 requires all exchanges to take a zero profit in pure-strategy equilibrium.

*Step 2: It is impossible for all exchanges to have zero profit in pure-strategy equilibrium.*

Suppose that all operators have zero profit. We can rank the feasible cum fee sell prices in equilibrium as $p_s^1 < p_s^2 < \cdots < p_s^Q$, where $Q$ is the number of feasible cum fee sell prices in all exchanges.

Similar to case (ii-b) in Proposition 3, two subcases are to be considered: (I) $p_b^1 = p_s^1 = a < \frac{1}{2}$; (II) $p_b^1 = p_s^1 = a \geq \frac{1}{2}$. [40] In (I), Operator $i$ can open a new EX using strategy (A.3) to ensure himself positive profits at positive probability. In (II), Operator $i$ can open a new EX using strategy (A.4) to ensure himself positive profits at positive probability.

Combining Steps 1–2, the proposition follows. ■

**Proof of PROPOSITION 7**

We prove the proposition using the example of two exchanges, and the results for multiple exchanges follow directly.

*Charging One Side Only:*

[40] Under multiple tick sizes, there are multiple cum fee sell prices that are determined by both the take fees and the liquidity maker's limit-order price. Such multiplicity does not affect the proof, because we need only a profitable deviation from the EX with the lowest cum fee sell price.

Here we consider the case in which only the liquidity taker is charged fees, and the liquidity maker is not charged, that is,

$$f_m^i = 0 \text{ and } f_t^i \geq 0.$$

The case in which $f_m^i \geq 0,$ and $f_t^i = 0$ can be proved similarly.

Recall that $\begin{cases} p_b^i = P^i + f_m^i \\ p_s^i = P^i - f_t^i \end{cases}$. When the rebate to the liquidity maker is banned, the liquidity maker's maximal surplus under $(0, f_t^i)$ becomes

$$BS^{i*}(v_b; f_t^i) = \max_{P^i}\{(v_b - P^i) \cdot \min\{1, 2(P^i - f_t^i)\} \big| v_b - P^i \geq 0 \text{ and } P^i - f_t^i \geq 0\}.$$

For any $f_t^1 \geq f_t^2 \geq 0$, we must have

(A.8) $$BS^{i*}(v_b; f_t^1) \leq BS^{i*}(v_b; f_t^2) \text{ for any } v_b.$$

Thus, any liquidity maker chooses the EX with the lowest take fee, if she chooses to trade. Then exchanges compete to undercut each other in take fee, which yields a Bertrand outcome.

*Equal Splitting the Total Fee:*

Now we consider the case in which the total fee is equally split between a liquidity maker and a liquidity taker, that is,

$$f_m^i = f_t^i = \frac{T^i}{2} \geq 0.$$

The liquidity maker's maximal surplus under $(\frac{T^i}{2}, \frac{T^i}{2})$ is

$$BS^{i*}(v_b; T^i)$$
$$= \max_{P^i}\left\{\left(v_b - P^i - \frac{T^i}{2}\right) \cdot \min\{1, 2\left(P^i - \frac{T^i}{2}\right)\} \middle| v_b - P^i - \frac{T^i}{2} \geq 0 \text{ and } P^i - \frac{T^i}{2} \geq 0\right\}.$$

For any $T^1 \geq T^2 \geq 0$, we must have

$$BS^{i*}(v_b; T^1) \leq BS^{i*}(v_b; T^2) \text{ for any } v_b.$$

Thus, any liquidity maker chooses the EX with the lowest $T^i$, if she chooses to trade. Then the competition between exchanges yields a Bertrand outcome. ■

# APPENDIX B: FEE STRUCTURE UNDER MULTIPLE TICKS

Here we present the optimal fee structure and profit as a function of $N$ when $N > 8$. When the value of $N$ is between 2 and 8, the optimal solutions are discussed on a case-by-case basis, the results of which are available upon request.

**PROPOSITION B1** (monopoly fee structure under multiple ticks). Under tick size constraints (1′), for $\mathrm{N} > 8$, the following fee structure maximizes the profit:

$$f_m^* = -\frac{\underline{n}}{N} + \frac{7N^2 + 8N + r \cdot (4 - r)}{16N^2}$$

$$f_t^* = \frac{\underline{n}}{N} - \frac{N^2 + 8N - r \cdot (4 - r)}{16N^2}$$

where $\frac{\underline{n}}{N}$ is the proposed limit-order price at $v_b = \frac{1}{2}$, i.e., $\underline{n} \equiv \left\lfloor \frac{N\left(\frac{1}{2} - f_m + f_t\right) + 1}{2} \right\rfloor$, and $r \in \{0,1,2,3\}$ is the remainder of $N$ divided by 4.

The maximum profit of the monopoly exchange is:

$$\pi^*(f_m^*, f_t^*) = \left[\frac{3}{8} + \frac{r \cdot (4 - r)}{8N^2}\right]^2,$$

$$\text{and} \lim_{N \to \infty} \pi^*(f_m^*, f_t^*) = \frac{9}{64}.$$

We prove this proposition in three steps.

**Step 1: Notations and Preliminary Analysis**

Denote the sum and difference between the make and take fees, respectively, as

$$T \equiv f_m + f_t, \text{ and } D \equiv f_m - f_t.$$

Note that the optimal make and take fees $(f_m, f_t)$ follow directly from $T$ and $D$.

A liquidity maker with valuation within $[\varphi_n, \varphi_{n+1}]$ proposes a limit order at price $P = \frac{n}{N}$. To find the threshold $\varphi_n$, we need

$$\left(\varphi_n - f_m - \frac{n-1}{N}\right)\left(-f_t + \frac{n-1}{N}\right) = \left(\varphi_n - f_m - \frac{n}{N}\right)\left(-f_t + \frac{n}{N}\right).$$

Thus

$$\varphi_n = f_m - f_t + \frac{2n-1}{N} = D + \frac{2n-1}{N}.$$

It follows that

$$\varphi_{n+1} - \varphi_n = \frac{2}{N}.$$

Therefore, the length of a complete interval $[\varphi_n, \varphi_{n+1}]$ for liquidity makers who propose the same limit-order price is $\frac{2}{N}$.

For now, we assume thatany liquidity maker with valuation $v_b \in [\frac{1}{2}, 1]$ participates in trading. We verify that such a condition is satisfied when $N > 8$. Full participation implies that the number of complete intervals contained in $[\frac{1}{2}, 1]$ is$\left\lfloor \frac{1-\frac{1}{2}}{\frac{2}{N}} \right\rfloor = \left\lfloor \frac{N}{4} \right\rfloor = \left\lfloor z + \frac{r}{4} \right\rfloor = z$, where $N = 4z + r$, $z$ is a non-negative integer, and $r \in \{0,1,2,3\}$ is the remainder of $N$ divided by 4.

We denote the number of thresholds $(\varphi_n)$ contained in $[\frac{1}{2}, 1]$ as $x$, noting that the thresholds closest to two ends of $[\frac{1}{2}, 1]$ may be from incomplete intervals. Then $x$ can only be $z$ or $z + 1$, as illustrated in Figure B1, where $\frac{n}{N}$ is the proposed price for $v_b = \frac{1}{2}$, i.e., $\underline{n} \equiv \left\lfloor \frac{N\left(\frac{1}{2} - f_m + f_t\right) + 1}{2} \right\rfloor$.

**Step 2: Optimal Fee Breakdown *D* for Given *T***

For any given $T$, suppose $D^* \equiv \underset{D}{\operatorname{argmax}}\, \pi(D; T)$. Then any disturbance $\Delta$ on $D^*$ such that $\widetilde{D} = D^* + 2\Delta$ (i.e., $\tilde{f}_m = f_m^* + \Delta$ and $\tilde{f}_t = f_t^* - \Delta$) should not improve profits. Under the new fee structure $(T, \widetilde{D})$ (or equivalently, $\tilde{f}_m = f_m^* + \Delta$ and $\tilde{f}_t = f_t^* - \Delta$), all thresholds have been shifted by $2\Delta$, that is,

$$\tilde{\varphi}_n = \varphi_n^* + 2\Delta.$$

There are two cases to be considered:

(i) $r \neq 0$

If $\Delta > 0$, all thresholds $\tilde{\varphi}_n$ shift toward the right, compared with $\varphi_n^*$. For each original interval $[\varphi_n^*, \varphi_{n+1}^*]$, $2\Delta$ liquidity makers reduce their quotes by one tick $(\frac{1}{N})$. As a result, the EX's execution probability for these liquidity makers decreases by $\frac{2}{N}$. Since there are $x$ thresholds, the aggregate profit loss for the EX is $T \cdot \frac{x \cdot 2\Delta}{1-\frac{1}{2}} \cdot \frac{2}{N} = 2\Delta T \cdot \frac{4x}{N}$.

At the same time, for the whole interval $[\frac{1}{2}, 1]$, a decrease in $\tilde{f}_t$ $\big($recall that $\tilde{f}_t = f_t^* - \Delta\big)$ increases the execution probability by $2\Delta$. It yields a profit gain for the EX of $T \cdot \frac{1-\frac{1}{2}}{1-\frac{1}{2}} \cdot 2\Delta = 2\Delta T$.

In panels (b) and (c) of Figure B1, there are $z$ effective thresholds when all thresholds $\tilde{\varphi}_n$ shift toward the right ($\Delta > 0$). Since $\frac{4x}{N} = \frac{4z}{N} < 1$, a deviation of $\Delta > 0$ increases the EX's total profit. So the cases presented in panels (b) and (c) cannot be optimal.

Similarly, the cases presented in panels (a) and (b) cannot be optimal, because $\Delta < 0$ leads to higher profit when $x = z + 1$.

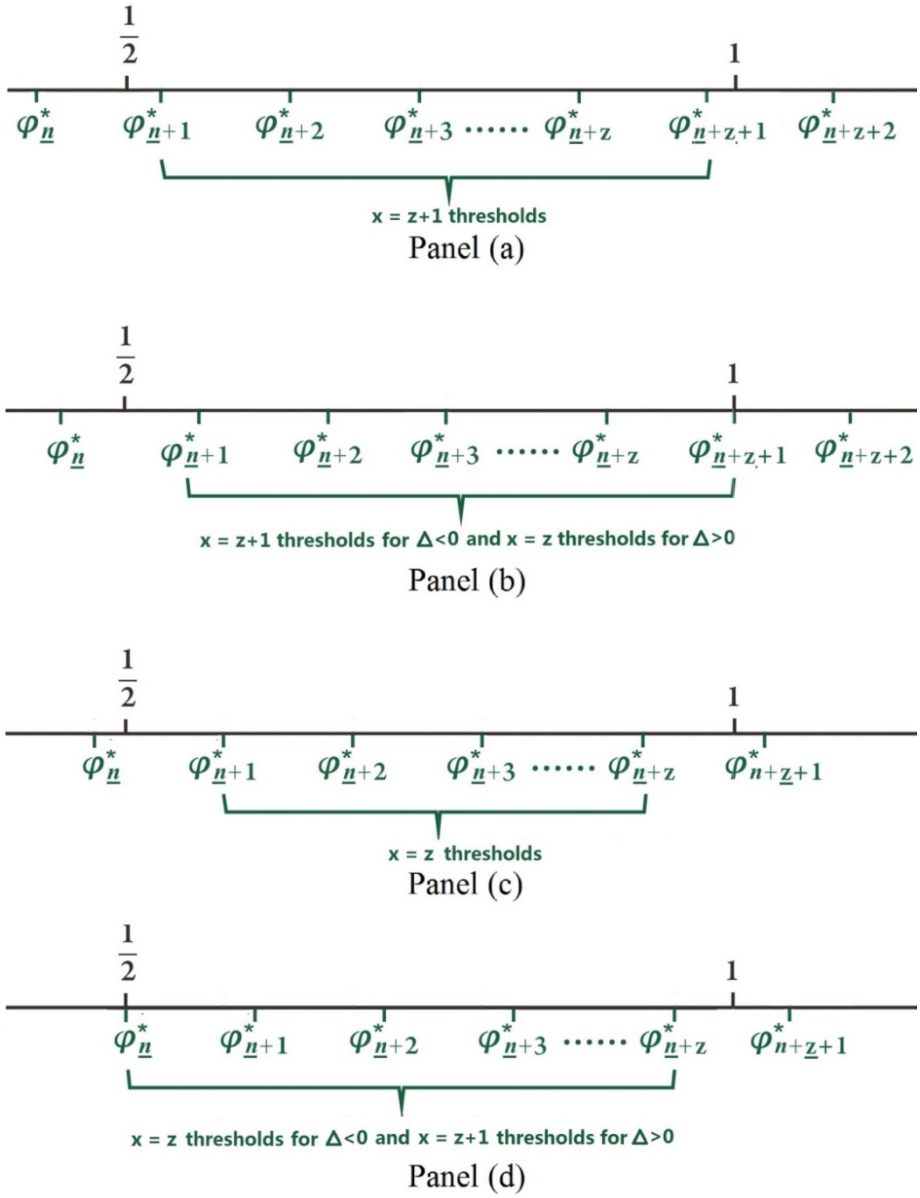


**FIGURE B1**

**Disturbance on Fee Breakdown**

As a result, only the case in panel (d) can be optimal. So at optimal, we have $\varphi_{\underline{n}} = \frac{1}{2}$.

(ii) $r = 0$

We have $x = z$ and thus $2\Delta T \cdot \frac{4x}{N} = 2\Delta T \cdot \frac{4z}{N} = 2\Delta T$, because $N = 4z$. Thus, a small disturbance $\Delta$ has no effect on the EX's profit. Therefore, fee breakdown happens to be neutral for a monopoly EX when $r = 0$. Without loss of generality, we assume $\varphi_{\underline{n}} = \frac{1}{2}$, so that we can use the same formula for any $N > 8$.[41]

Combining (i) and (ii) above, at optimum, we must have $\underline{n} = \left\lfloor \frac{N\left(\frac{1}{2} - f_m + f_t\right) + 1}{2} \right\rfloor$ such that $\varphi_{\underline{n}} = D + \frac{2\underline{n}-1}{N} = \frac{1}{2}$. This condition implies that $D = \frac{1}{2} - \frac{2\underline{n}-1}{N}$. It follows that

$$f_m = \frac{T+D}{2} = \frac{T - \frac{2\underline{n}-1}{N} + \frac{1}{2}}{2}, \qquad f_t = \frac{T-D}{2} = \frac{T + \frac{2\underline{n}-1}{N} - \frac{1}{2}}{2}.$$

**Step 3: Determination of Optimal *T***

When any liquidity maker with valuation $v_b \in [\frac{1}{2}, 1]$ submits a limit order, the EX's expected profit depends on the total fee and the limit order's execution probability.

$$\pi(f_m, f_t) = \text{total fee} \times \sum_{\text{all price levels}} (\text{execution probability} \times \text{interval width})$$

$$= T \cdot 2 \cdot \left[ \sum_{i=\underline{n}}^{z+\underline{n}-1} \left(-f_t + \frac{i}{N}\right) \cdot \frac{\frac{2}{N}}{1-\frac{1}{2}} + \left(-f_t + \frac{z+\underline{n}}{N}\right) \cdot \frac{\frac{r}{2N}}{1-\frac{1}{2}} \right]$$

$$= T \cdot \left[\frac{3}{4} + \frac{r \cdot (4-r)}{4N^2} - T\right]$$

Thus,

$$T^* = \frac{3}{8} + \frac{r \cdot (4-r)}{8N^2}, \text{ and } \pi^* = \left[\frac{3}{8} + \frac{r \cdot (4-r)}{8N^2}\right]^2.$$

The optimal fees follow directly as

$$f_m^* = \frac{T^* + D^*}{2} = -\frac{\underline{n}}{N} + \frac{7N^2 + 8N + r \cdot (4-r)}{16N^2}$$

$$f_t^* = \frac{T^* - D^*}{2} = \frac{\underline{n}}{N} - \frac{N^2 + 8N - r \cdot (4-r)}{16N^2}.$$

This proof requires the full participation of liquidity makers even after we increase the make fee by an amount $\Delta > 0$. This full participation requires the liquidity maker with

[41] Yet we are aware that, for $r = 0$, Proposition B1 only presents one set of fees that maximize the profit. When N is the multiple of 4, the fee breakdown happens to be neutral and the optimal fee and profit coincides with the case of zero tick size under one (monopoly) exchange. Yet Table II shows that fee breakdown is no longer neutral when the monopoly operator establishes two exchanges, and Proposition 6 shows that no pure strategy exists for any tick size *N*.

valuation $v_b = \frac{1}{2}$ to quote a limit-order price such that the cum fee sell price is larger than $\frac{1}{N}$. If not, an increase in make fee leads the liquidity maker with $v_b = \frac{1}{2}$ to quote a price one tick lower because she is at the threshold, which leads to a cum fee sell price no larger than zero. Because no liquidity taker accepts a negative cum fee sell price, the full participation condition is violated. Therefore, we need

$$p_s^* = -f_t^* + \frac{n}{N} = \frac{N^2 + 8N - r \cdot (4 - r)}{16N^2} > \frac{1}{N},$$

which reduces to $N > 8$. ■

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**TABLE I**

**Participations in Exchange(s) Established by a Monopoly and the Welfare of the Three Parties**

| | One Exchange | Two Exchanges | |
|---|---|---|---|
| | | Low Quality | High Quality |
| **Panel A: Cum Fee Prices** | | | |
| *Cum Fee Sell Price* | 1/3 | 1/5 | 2/5 |
| *Execution Probability (Quality)* | 2/3 | 2/5 | 4/5 |
| *Cum Fee Buy Price* | 2/3 | 3/5 | 7/10 |
| **Panel B: Range of Participants** | | | |
| *Liquidity Maker Participation* | [2/3, 1] | [3/5, 4/5] | [4/5, 1] |
| *Liquidity Taker Participation* | [0,1/3] | [0,1/5] | [1/5, 2/5] |
| **Panel C: Welfare** | | | |
| *Operator* | 4/27 | 4/25 | |
| *Liquidity Maker* | 2/27 | 2/25 | |
| *Liquidity Taker* | 2/27 | 2/25 | |

*Notes.* This table shows the market outcomes when a monopoly establishes one exchange and when a monopoly establishes two exchanges. Panel A displays the cum fee sell and buy prices in each exchange. Panel B displays the valuation ranges of liquidity makers and liquidity takers who participate in the exchange, and their corresponding choices of the exchanges. Panel C shows the expected welfare for the operator, liquidity makers, and liquidity takers.

**TABLE II**

**Exchange Fee Structures and Liquidity Makers' Segmentation under Various Tick Sizes**

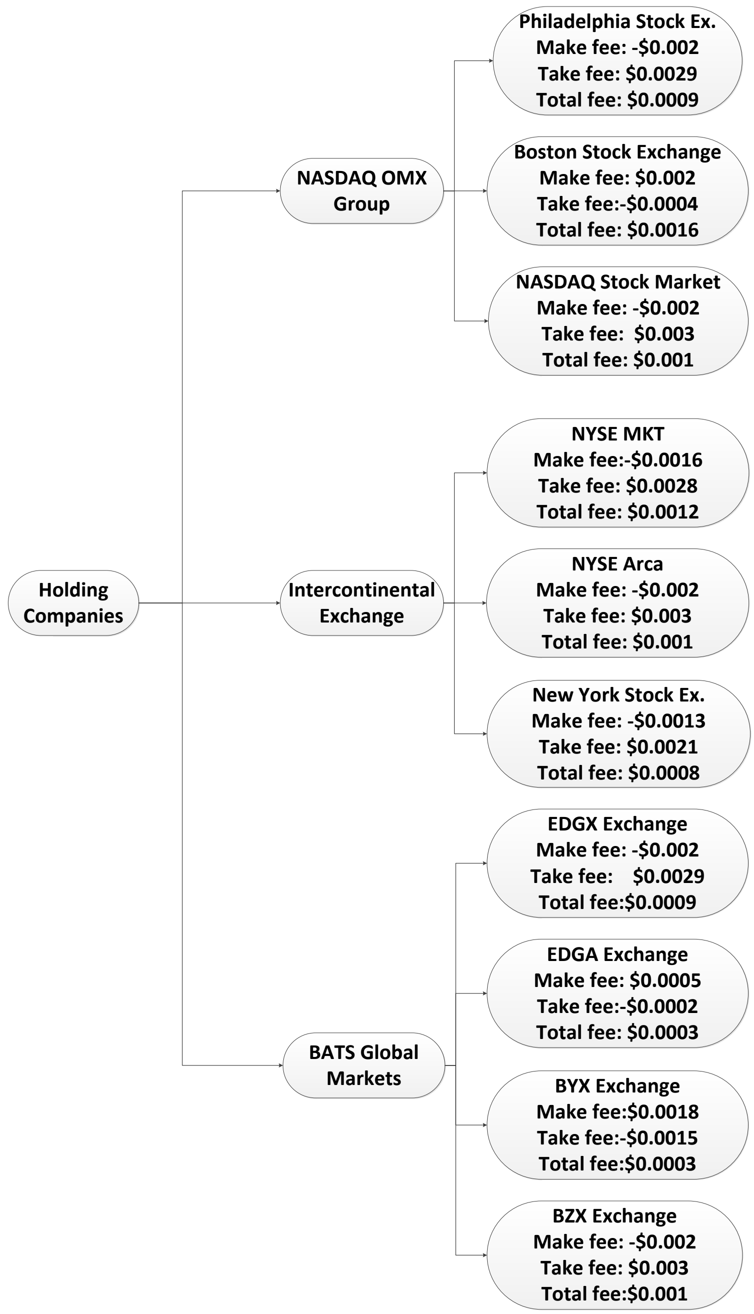
| | $N$ = 2 | | | $N$ = 4 | | | $N$ = 8 | | |
|---|---|---|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
| | One Exchange | Two Exchanges | | One Exchange | Two Exchanges | | One Exchange | Two Exchanges | |
| | | Exchange 1 | Exchange 2 | | Exchange 1 | Exchange 2 | | Exchange 1 | Exchange 2 |
| **Panel A: Fee Structure of Exchanges** | | | | | | | | | |
| *Make Fee* | 0.6667 | 0.6000 | 0.7000 | 0.3750 | 0.4095 | 0.5840 | 0.3750 | 0.4165 | 0.4705 |
| *Take Fee* | -0.3333 | -0.2000 | -0.4000 | 0.0000 | -0.0340 | -0.1595 | 0.0000 | -0.0339 | -0.0961 |
| *Total Fee* | 0.3333 | 0.4000 | 0.3000 | 0.3750 | 0.3755 | 0.4245 | 0.3750 | 0.3826 | 0.3744 |
| *Profit* | 0.1481 | 0.0640 | 0.0960 | 0.1406 | 0.1107 | 0.0338 | 0.1406 | 0.0421 | 0.0994 |
| **Panel B: liquidity Markers' Segmentation** | | | | | | | | | |
| $P$=0 | 0.6667–1.0000 | 0.6000–0.8000 | 0.8000–1.0000 | | 0.5000–0.6313 | 0.6313–0.7562 | | | 0.5000–0.6501 |
| $P$=1/8 | | | | | | | 0.5000–0.7500 | 0.6501–0.7334 | 0.7334–0.9164 |
| $P$=2/8 | | | | 0.6250–1.0000 | 0.7562–1.0000 | | 0.7500–1.0000 | 0.9164–1.0000 | |
| $P$=3/8 | | | | | | | | | |
| $P$=4/8 | | | | | | | | | |

*Notes.* This table provides an example of fee structures set by a monopoly and the segmentation of liquidity makers under various tick sizes. Columns (1)–(3), (4)–(6) and (7)–(9) display the market outcomes under tick sizes of $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$, respectively. Columns (1), (4), and (7) display cases in which the monopoly establishes only one exchange, and columns (2)–(3), (5)–(6), and (8)–(9) display cases in which the monopoly establishes two exchanges. Panel A lists the fee structures chosen by the monopoly. The horizontal row in Panel B indicates the limit-order price that the liquidity maker proposes and the vertical row indicates the liquidity maker's choice of exchange conditional on her valuation of the security presented in the cell.

## FIGURE I

### Structure of U.S. Stock Exchanges and Corresponding Fee Structures on Each Exchange

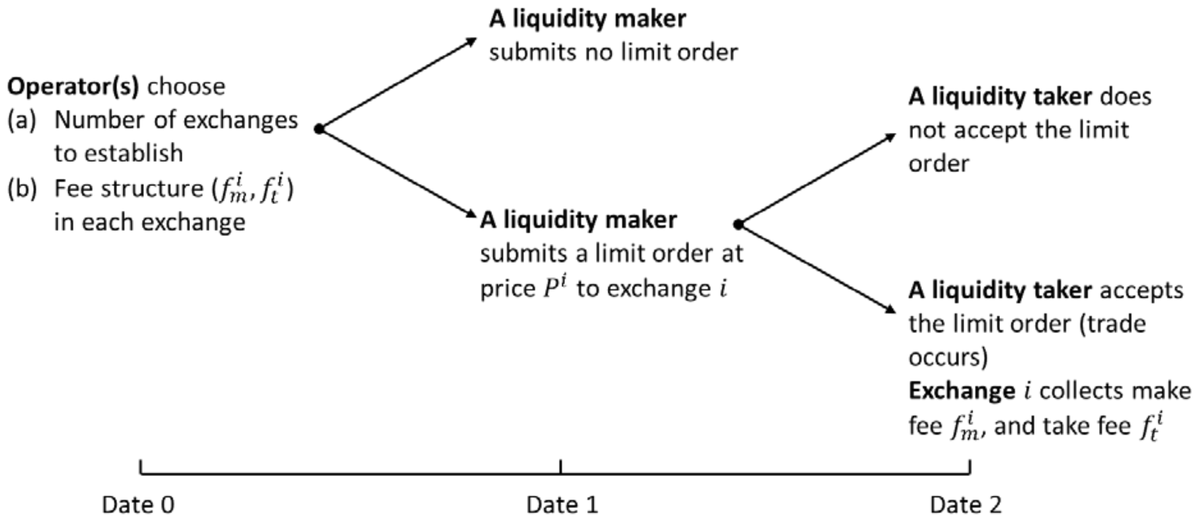
*Notes.* This figure displays the ten U.S. stock exchanges run by three holding companies (operators). This figure also displays the corresponding make/take fees on each exchange in May 2015.[42]




[42] The figure displays the fee structure for NYSE-listed stocks. Standard rates are presented in the figure; for exchanges not stating standard rates, fees unconditional on the participants' activities are presented.

### FIGURE II

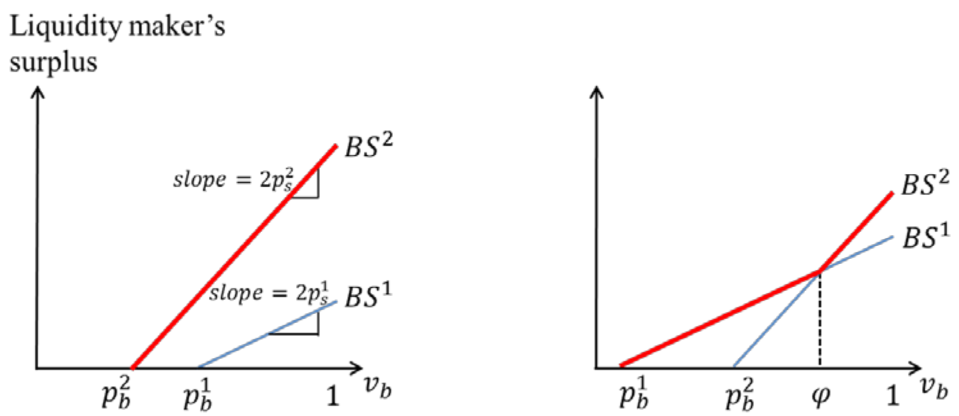
## Timeline of the Model



## FIGURE III

**Liquidity Maker's Surplus under Two Exchanges**

*Notes.* This figure shows the liquidity maker's surplus when choosing between two exchanges. The horizontal axis indicates the liquidity maker's valuation of the security and the vertical axis indicates the liquidity maker's trading surplus. Without loss of generality, we assume that the cum fee sell price in Exchange 1, $p_s^1$, is lower than the cum fee sell price in Exchange 2 , $p_s^2$. $BS^1$ and $BS^2$ depict the liquidity maker's surplus when choosing Exchange 1 and Exchange 2, respectively. The liquidity maker chooses the exchange that offers her a higher surplus. The thick red lines depict the liquidity maker's choice, which is the upper envelope of the two surplus curves. $\varphi$ is the cut-off value for the marginal liquidity maker.




Panel (a) $p_b^2 \leq p_b^1$
Exchange 2 Only

Panel (b) $p_b^1 < p_b^2$
Exchange 1 and 2 Co-exist if $\frac{1}{2} < \varphi < 1$

**FIGURE IV**

**Two Types of Deviation from Bertrand Equilibrium**

*Notes.* This figure shows two types of profitable deviation for Exchange 2 when Exchange 1 and Exchange 2 start by setting the total fee at zero. The horizontal axis indicates the liquidity maker's valuation of the security and the vertical axis indicates the liquidity maker's trading surplus. $BS^1$ and $BS^2$ depict the liquidity maker's surplus when choosing Exchange 1 and Exchange 2, respectively. In Panel (a), exchange 1 and exchange 2 start by setting $p_b^1 = p_s^1 = \frac{1}{2}$. Exchange 2 can profitably deviate by decreasing the cum fee buy price to $p_b^2 = \frac{1}{2} - \mu\varepsilon$ and the cum fee sell price to $p_s^2 = \frac{1}{2} - \varepsilon$. In Panel (b), Exchange 1 and Exchange 2 start by setting $p_b^1 = p_s^1 < \frac{1}{2}$. Exchange 2 can profitably deviate by increasing the cum fee buy price to $p_b^2 = p_b^1 + \varepsilon$ and the cum fee sell price to $p_s^2 = p_s^1 + \mu \cdot \varepsilon$. $\varepsilon > 0$ and $0 < \mu < 1$ in both panels. The liquidity maker chooses the exchange that offers her a higher surplus. The thick red lines depict the segment of liquidity makers who are drawn to Exchange 2. $\varphi$ is the cut-off value for the marginal liquidity maker.