Dynamic Compensation Contracts with Private Savings

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Abstract

This paper studies a dynamic agency problem where a risk-averse agent can privately save. In the optimal contract, wages are downward rigid; permanent pay raises occur when the agent's historical performance is sufficiently good; and when the agent is dismissed due to his poor performance, he walks away with a severance pay to support his post-firing consumption at the current wage level. Thus, under realistic contracting frictions, seemingly inefficient compensation schemes can indeed be optimal. Several extensions are considered, including renegotiation-proof contracts.

Key Words: Continuous-time Contracting, Poisson Process, Wealth Effect, Cost of High-Powered Incentives.

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1 Introduction

This paper studies a dynamic agency problem where the risk-averse agent controls the firm's profitability through unobservable actions, and he can privately save (or, hidden savings). We find that in the optimal compensation contract, wages are non-decreasing, and the agent gets a severance pay even when he is dismissed due to his poor performance. Both features are commonly observed in reality, though often times they are accused as symptoms of corrupted corporate governance and inefficiency in executive compensation.

In the dynamic contracting literature, starting from Rogerson (1985) and among others, researchers have found that without private savings, the optimal wage pattern tends to be front-loaded (or, an inverse-martingale property, under which the agent's expected marginal utility from consumption increases over time). Therefore, the agent tends to smooth his consumption via the private savings account, which will devastate the incentive scheme designed in the optimal contract. Consistent with this tension, the optimal contract in this paper features a back-loaded, non-decreasing wage pattern, which precludes the agent's motive to engage in private saving.

The general optimal contracting problem with private savings is complicated; we solve this model under a specific setting. In this paper, cashflows follow a Poisson process. The cashflow arrival intensity is controlled by the agent's three levels of unobservable effort (action): *null, normal,* and *myopic*, and the optimal contract implements the interior normal effort. The null effort (as *shirking*) leads to no cashflow in the next time interval, and the normal effort (as *working*) generates a positive success intensity. The myopic action—borrowing the spirit from Stein (1989) and Holmstrom and Milgrom (1991)—helps to improve the short-term "hard" cashflow performance; but this action either hurts the firm's long-run value, or undermines the agent's other "soft" performances that are critical to the firm.

We envision that these long-run destructions, either are realized after the agent's tenure, or take forms unforeseen by investors, are not contractible. This just captures the cost of high-powered incentive schemes, a well-documented economic phenomenon (e.g., Levitt and Dubner (2005), Larkin (2006)).[1] In

[1]For a nice review, see Larkin (2006). Citing from that paper, perhaps the most celebrated example is Sear's experience offering commissions to its auto mechanics based on total charges for parts and labor. Mechanics responded to this scheme by ordering unneeded repairs, and Sears ended up settling a class-action lawsuit over excessive billing. This greatly damaged the

corporate finance, this idea is connected to Stein (1989) and the follow-up papers, and the ongoing literature on the over-valued equity and related agency issues (e.g., Jensen (2005))). In contrast to Stein (1989), in our model, the optimal contract discourages the agent's myopic behaviors by avoiding excess incentive loadings on the short-term cashflow performance.

This finding, together with the linear effort cost structure, implies that investors should provide exact working incentives for the agent, and a binding incentive-compatibility constraint (with respect to effort deviations only) is key to solving the contracting problem with private savings. In fact, the "punishing" pattern of decreasing wages following unsatisfactory performance, is not viable for a contract to implement the interior working effort when the agent can privately save. The argument is based on the agent's potential "joint deviations" detailed in Section 3. In words, a binding incentive-compatibility constraint implies that the agent is indifferent between shirking and working, *holding the equilibrium consumption plans fixed*. Now consider the joint-deviation strategy of "shirking and saving;" because shirking leads to a zero success probability, if the contract assigns decreasing wages for no success, then the agent can strictly improve his payoff by smoothing his consumption on the path of no success. As a result, the optimal contract features a non-decreasing wage given no success.

In Section 4, we solve the dynamic contracting problem based on two state variables: the agent's marginal utility (which captures his current wage as suggested above), and the agent's continuation payoff. In the optimal contract, wages are downward rigid; the agent is guaranteed with the current wage level, and works for future pay raises (as promotions). Also, when a streak of poor performance leads to an endogenous termination (firing), the agent walks away with a severance pay to support his post-firing perpetuity consumption at the current wage level, and the severance pay is increasing in his past performance. These features are widely observed in practice.

Whether private savings are possible or not makes the optimal wage contract drastically different. Figure 1 compares the optimal wage process in our model, to the one derived under the same setup except that the agent's savings are observable. In the figure, for both cases, the agent starts with the same initial state, and experiences the same cashflow performance (at $t = 0.5, 1, 1.5,$ and 3.5). When savings are observ-

reputation of Sears' car mechanic arm.

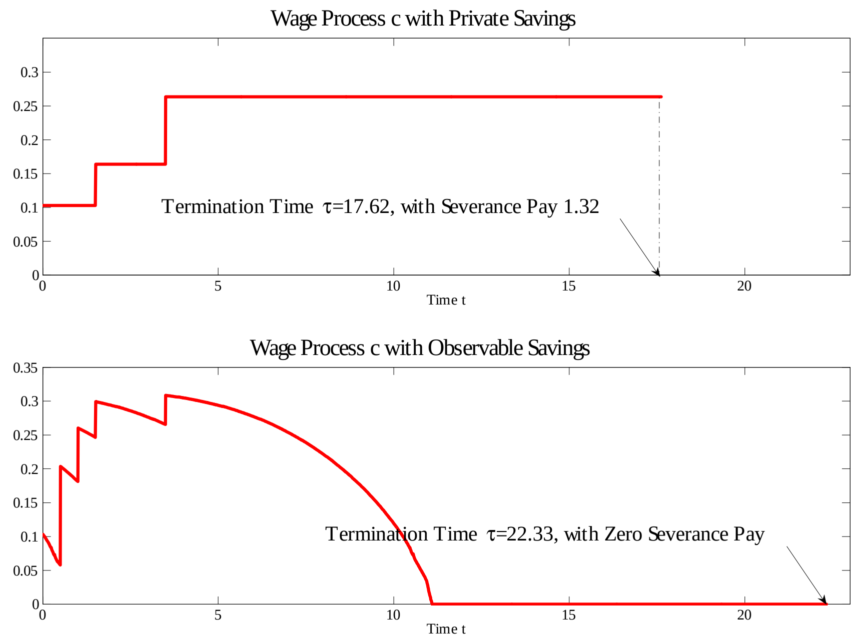

Figure 1: Optimal wage policies with private savings (top panel) and observable savings (bottom panel).

able, the agent's wages display a "zig-zag" pattern, and respond actively to not only *cashflow realizations*, but also *no cashflow realizations* (which represent poor performance in this model). In contrast, when the agent can privately save, wages are only adjusted upward, and never respond to poor performance. In addition, in the case with observable savings, the fired agent leaves the company with nothing; however, the private-saving case features a positive severance pay when the agent is dismissed due to his poor performance.[2]

Several extensions are considered in Section 5, including the renegotiation-proof contract, and the possibility of agent's unobservable initial wealth. We also show that our model is robust to the non-contractibility of the myopic action loss. In Section 5.3, motivated by a multi-tasking model analyzed in Holmstrom and Milgrom (1991), we allow this non-contractible loss to be reflected by a noisy (to capture its long-run and soft nature) but contractible measure. When the precision of this noisy measure goes to

[2]There are other papers show that positive severance payments, by soothing the agent's fear of dismissal, might provide proper incentives for risk-taking (Berkovitch, Israel, and Spiegel (2000)), or complete information disclosure (Inderst and Mueller (2005)). In essence, these findings are along the same line as this paper: by promising a generous severance pay, the contract prevents the agent from harmful deviation strategies (e.g., shirking and saving in this paper; see Section 3.2).

zero, the value from a complete contract, which incorporates the soft measure, converges to the one from the *incomplete* contract that simply ignores this almost informationless measure. Therefore, if there exists some fixed information acquisition cost, then the contract derived in this paper could be indeed optimal in a *complete* contract paradigm.

This paper belongs to the burgeoning continuous-time contracting literature. DeMarzo and Sannikov (2006) study a continuous-time version of the DeMarzo and Fishman (2006) model; Biais et al. (2007) obtain similar results.[3] In all three papers the agent is risk-neutral, which eliminates the saving incentives. Sannikov (2006) studies an optimal contracting problem with a risk-averse agent; but in that paper the agent's savings are observable.

Under the similar observation of implementing an interior effort, Kocherlakota (2004) solves an optimal unemployment insurance contract where the agent's savings are private. That model features a single success when the agent becomes permanently employed. In contrast, this paper studies a more general setup with multiple cashflows, and allows for endogenous termination (firing) in the employment contract. Moreover, our corporate finance framework allows us to introduce the agent's myopic behaviors into the model, and based on this we provide justifications and proofs for the optimality of the contract.[4]

Harris and Holmstrom (1982) find that the downward-rigid wage is optimal. Their mechanism is fundamentally different from ours. In their model, without moral hazard problems, the first-best wage contract features a constant wage for the risk-averse agent to fully insure his productivity shocks. If the agent can quit, then a competitive labor market implies that looking forward, the agent's future compensation has to stay above his expected productivity at any time during the employment. As a result, the agent's ex-post participation constraint might be binding, and to match the agent's outside option the contract will specify

[3] To name a few follow-up studies, in an application to executive compensation, He (2007a) extends the analysis to the geometric Brownian motion model; and Piskorski and Tchistyi (2007) study the optimal mortgage design by considering the exogenous regime switching in the investors' discount rate. Another strand of continuous-time contracting literature starts from Holmstrom and Milgrom (1987). In fact, in their model, the CARA preference with monetary effort cost is free of the wealth effect, which greatly facilitates the contracting problem with private savings. Under that framework, allowing for negative consumption and two-way transfer between the principal and the agent is important. See He (2007b) and Williams (2006).

[4] A recent paper by Mitchell and Zhang (2007) shows that it is never optimal to implement interior effort in the setting of Kocherlakota (2004). Similar to Kocherlakota (2004), they only consider one single success, and there is no termination. Interestingly, based on CARA preference and *linear additive effort costs* (rather than *monetary effort costs* as in Holmstrom and Milgrom (1987)), they provide a nice solution to optimal contracting with private savings and binary-effort; in their analysis, allowing for negative consumption is important.

a wage raise in response to sufficiently good news about the agent's productivity.[5]

Atkeson and Cole (2005) analyze a dynamic version of the costly-state-verification model, and show that the optimal wage is also non-decreasing. Compared to this paper, they rule out private savings when the agent mis-reports, and the agency issue in their paper is distinct from moral hazard in the nature. In the costly-state-verification framework, a back-loaded compensation, by strengthening the potential punishment, can alleviate the agency problem. In contrast, under moral hazard as studied in this paper, a back-loaded compensation potentially exacerbates the agency issue through the wealth effect.[6]

The rest of this paper is organized as follows. Section 2 describes the model. Section 3 gives two key state variables in solving the optimal contract recursively, and Section 4 shows that downward rigid wages with severance pay is optimal. Section 5 considers various extensions, and we conclude in Section 6. All proofs are given in the Appendix.

2 The Model

2.1 Technology

Consider a continuous-time principal-agent model, where the risk-neutral investors (the principal) of an infinitely-lived firm hire a risk-averse agent for business operation. The firm generates cashflows YdN_t at each instant of time, where $\{N_t\}$ is a standard Poisson process with intensity $\{a\}$, and Y is a positive constant. Later on we use "cashflow," "jump," and "success" interchangeably. The cashflows are observable and contractible. As we will discuss shortly, the firm may have other value-generating businesses.

The agent can generate at most K cashflows; later we mainly focus on the stationary case where $K \to \infty$. When the employment ceases, investors can liquidate the firm's assets for an exogenous value L, which is normalized to zero. One can easily endogenize L by a costly replacement with another new agent. Both the agent and investors discount future payoffs at a constant market interest rate $r > 0$.

[5]A recent paper by Berk, Stanton, and Zechner (2007) embeds a capital structure decision into this framework; they highlight the importance of human capital when firms choose the optimal capital structure. The resulting wage contract is not downward rigid. Therefore, even though the optimal contract in Harris and Holmstrom (1982) is private-saving-proof because of the downward-rigidity, the one derived in Berk et al. (2007) is not.

[6]Also, Atkeson and Cole (2005) derive the properties of the optimal contract based on the interior first-order conditions, without verifying that the firm's value function—under the optimal policy—is indeed concave. In fact, as shown in Section 5.4 in this paper, a non-concave value function calls for value-improving randomizations, which can affect the associated optimal policy. For instance, the optimal wage process, along the optimal path with lotteries (mean-preserving randomizations), fails to be non-decreasing.

The agent's unobservable effort a controls the intensity process. There are *three* effort levels, i.e., $a \in \{0, p, \overline{p}\}$ where $\overline{p} > p > 0$. Agent's non-pecuniary personal effort cost (or private benefit if negative) when exerting a, in terms of the agent's utilities, is $b\left(\frac{a}{p} - 1\right) dt$, where b is a positive constant. We call the lowest effort $a = 0$ the *null* effort (or, *shirking*). By shirking, the agent enjoys a private benefit bdt, but the intensity of cashflow is zero. The agent can choose the median *normal* effort $a = p$ (or, *working*); in this case he obtains no private benefit, but the firm generates cashflows with a probability pdt.

Myopic Action and Its Non-contractible Loss In this model, the agent can exert the highest *myopic* effort $\overline{p} = p + \epsilon$ to increase the cashflow intensity. In the spirit of Stein (1989), this myopic action is detrimental, because it represents the short-term performance-enhancing strategies that hurt either the firm's long-run value, or other dimensions of soft performances that are essential to the firm. We assume that these losses borne by investors, Δdt, are non-contractible.

We can model this in the following way. Assume that the liquidation value L is positive and random; and whenever the agent exerts $a = \overline{p}$, the expected (discounted) liquidation value L drops by at least Δdt. During termination, the investor (as a bank who has specialty to locate the second-best users) handles the liquidation process, and reports a liquidation value $\widehat{L}$ which might differ from the true liquidation value L. Ruling out a third party (due to the possibility of collusion, etc.), the information revealed by the report $\widehat{L}$ becomes as if non-contractible.[7]

However, there are other ways to interpret this non-contractible loss due to the agent's myopic actions, and in this paper we will keep our interpretations general. For instance, transforming the physical machine to an asset special to the agent (Shleifer and Vishny (1989)), or reputation-destroying accounting scandals that are uncovered after the agent's tenure, shares the same spirit as myopic behaviors.[8] It also relates to

[7]Two points are note-worthy. First, in fact as argued in Hart and Moore (1998) footnote 4, if investors value these liquidated assets more than the market does, then the liquidation value can be non-verifiable, therefore non-contractible. Second, when $K = \infty$ (recall K is the maximum number of cashflows that an agent can generate) as in our later analysis, it is possible that along certain equilibrium path (the first-best absorbing states), the future termination probability becomes zero. However, for any finite K, the discounted probability of termination is uniformly strictly positive, and investors always put certain positive weights on the loss (on L) due to myopic actions. Since our analysis applies to any finite K, the results in this paper can be viewed as the limiting case of the K-cashflow model.

[8]For instance, in August 2007, Dell restated down its past four years' earnings by up to $150 million, and the executives who were responsible to this scandal have left the company; data source: "Dell to Restate Earnings After Probe," http:biz.yahoo.com/ap/070816/dell_restatement.html. Notice that the agent's higher personal cost due to myopic actions could incorporate the future extra cost borne by himself (such as career concerns), if any; the key is that they are not within the current

the multi-tasking problem studied in Holmstrom and Milgrom (1991); there, if the compensation contract imposes excessive incentives on the hard and easy-to-measure performance (cashflow occurrence in this model), the agent will ignore other dimensions of soft performances that are critical to the firm—for instance, refusing to collaborate with his colleagues thereby lowering their efficiency, creating a harsh working environment that is less attractive to young talents, and so on. Clearly these actions, and the consequences brought on by them, are extremely difficult to verify ex-post and contract ex-ante.

In sum, the non-contractible loss due to the myopic action captures the cost of high-powered incentive schemes, a well-documented fact in both economic and finance literatures.[9] Once equipped with excessive incentives, the agent will be motivated to drive up short-term performance, but hurt the firm in certain ways that investors either cannot specify the damages in some verifiable terms ex-ante,[10] or are only able to discover future losses after the agent's tenure. As a result, investors are averse to deliver high-powered incentives during employment, and myopic actions are off-equilibrium in this model (which differs from Stein (1989) where the manager takes myopic actions in equilibrium).

Throughout the paper we consider the case that it is optimal to implement the normal working effort $a = p$. We verify the optimality of this policy later in Section 4.3.2. We should add that the discrete structure of the agent's action space is immaterial; the key is the linearity of the agent's effort cost structure, and implementing the interior effort.

employment contract.

[9]For a real-world example, see footnote 1. The literature starting from Stein (1989) studies managers' myopic behaviors once they are concerned about the firm's stock price. Along the ongoing literature of agency costs due to overvalue equity (Jensen (2005)), Efendi et al. (2007) find that executives are more likely to misstate financial statements when they have sizable amount of stock options "in-the-money," and exercise more options during the first year of restatement. Kothari et al. (2005) attribute the negative relation between accounting accruals and subsequent returns to the manager's higher misreporting incentive for overvalued equities.

[10]As a criticism to the research on property-rights in the incomplete contracting literature, Maskin and Tirole (1999) point out that the ex-ante indescribability can be fully overcome, *if realized states can be costlessly elicited ex-post from both parties though an implementation mechanism*. The sufficient condition for this qualification–the so-called welfare-neutrality condition—holds in the standard property-rights model, but fails in the moral-hazard problem studied here. See Maskin and Tirole (1999) for details.

2.2 The Agent

Utility Function The agent's utility from consumption is $u(c_t)\,dt$, where $u' > 0$, $u'' \leq 0$, and $c_t \geq 0$ is the consumption rate. The agent's total utility, $\widetilde{u}(c_t, a_t)$, takes an additive form, i.e.,

$$\widetilde{u}(c_t, a_t) = u(c_t) + b\left(1 - \frac{a_t}{p}\right). \tag{1}$$

By taking the above additive form, the wealth effect is always present. For the normal working effort to be optimal all the time, we have to rule out the "extreme" wealth effect. Formally, there exists a strictly positive number $\underline{\gamma}$ such that

$$\inf_{c \geq 0} u'(c) = \underline{\gamma} > 0. \tag{2}$$

To see this, from the agent's view, the monetary equivalence (marginally) of the effort cost is b/u'. Therefore, by focusing on the moderate wealth effect, essentially (2) places an *upper* bound on the agent's monetary effort cost. This assumption captures the idea that the firm has the option to adopt a monitoring technology at a high but bounded cost. In addition, given a finite number (K) of cashflow jumps, the marginal utility level $\underline{\gamma}$ may not be reached in equilibrium. Finally, condition (2) is for tractability reasons only, and we can numerically solve the optimal contract for the $\underline{\gamma} = 0$ case (see Section 5.4).

Though our results hold for general utilities, in the main analysis we focus on one special form of $u(\cdot)$, which is the *modified* CARA (Constant Absolute Risk Aversion) utility defined as follows:

$$u(c) = \begin{cases} 1 - e^{-\gamma c} & \text{if } c < \frac{1}{\gamma}\ln\frac{\gamma}{\underline{\gamma}} \\ 1 - \frac{\underline{\gamma}}{\gamma} + \underline{\gamma}\left(c - \frac{1}{\gamma}\ln\frac{\gamma}{\underline{\gamma}}\right) & \text{otherwise} \end{cases}. \tag{3}$$

In words, to respect condition (2), we simply replace the upper part (when $c \geq \frac{1}{\gamma}\ln\frac{\gamma}{\underline{\gamma}}$) of the CARA utility with a linear function with a slope $\underline{\gamma}$ (so the agent becomes risk-neutral when he is sufficiently wealthy). The CARA form possesses the convenient feature that the *marginal utility* is linear in the *utility* level, which simplifies the analysis greatly. However, as shown in Section 5.1, our optimal contracting results do not depend on the CARA form.

Private Savings In this paper the agent can privately save for the consumption smoothing purpose. As first noted by Rogerson (1985), when the agent's utility is additive as in (1), the optimal contract without private savings features an "inverse-martingale property" (or, "*compensation* smoothing property" in Sannikov (2006)). Under this property, the agent's marginal utility follows a submartingale, and the salary pattern is front-loaded. Clearly this is against the "*consumption* smoothing property" if the agent can privately save.

We rule out the agent's borrowing from a third party. Clearly, the borrowing technology where a bank expects repayments is somewhat inconsistent with the agents' private-saving technology. In fact, the CARA preference (with monetary effort cost) with borrowing and negative consumption allows for a tractable solution with private savings (see Williams (2006), and He (2007b)).

2.3 Contract

An employment contract specifies a wage process $\{c_t \geq 0 : 0 \leq t < \tau\}$, and a lump-sum transfer $F_\tau \geq 0$ at the termination event (when the agent is fired), where τ is the endogenous termination time (which might be before the K^{th} cashflow realization). We denote such a contract as $\Pi \equiv \{\{c\}, F_\tau, \tau\}$, and each element is N-measurable, i.e., the "hard" cashflows are the only contractible performance. Here, because of the agent's limited liability, any contractual payment to the agent must be nonnegative.

The agent has zero initial wealth (see Section 5.5 for a discussion of unobservable initial wealth); and for simplicity, after τ the agent remains unemployed forever (so his outside option is zero). Denote the agent's savings account (with an interest rate r) balance as $S_t \geq 0$ (recall the borrowing constraint). Given the contract Π, the agent's problem is

$$\begin{aligned} \max_{\{a\},\{\widehat{c}\},\widehat{c}_\tau} \quad & \mathbb{E}^a \left[\int_0^\tau e^{-rt} \left[u(\widehat{c}_t) - \frac{b}{p}(a_t - p) \right] dt + e^{-r\tau} \frac{u(\widehat{c}_\tau)}{r} \right] \\ s.t. \quad & dS_t = rS_t dt + c_t dt - \widehat{c}_t dt \text{ with } S_0 = 0,\, S_t \geq 0 \text{ for } 0 \leq t \leq \tau \\ & \widehat{c}_\tau = r\left(F_\tau + \widehat{S}_\tau\right), \end{aligned} \tag{4}$$

where $\mathbb{E}^a[\cdot]$ indicates that the probability measure is induced by the agent's effort policy $\{a\}$, and $\{\widehat{c}\}$ and $\widehat{c}_\tau$ are privately observed consumptions during and after the employment, respectively. Note that the concavity of u implies a constant consumption level $\widehat{c}_\tau$ in the agent's post-firing life, and $e^{-r\tau}\frac{u(\widehat{c}_\tau)}{r}$ captures the (total) discounted utility after the termination.

According to the revelation principle, we can restrict our attention to the case with $S_t = 0$. Intuitively, whenever the agent wants to save, investors can do the savings for him. Therefore, we call the contract Π incentive-compatible and no-savings, if $\{\{p\}, \{c\}, rF_\tau\}$ solves the problem (4).

The optimal contract solves the investors' problem:

$$\max_{\Pi \text{ is incentive-compactible and no-savings}} \mathbb{E}\left[\int_0^\tau e^{-rt}\left(YdN_t - c_t\right)dt - e^{-r\tau}F_\tau\right],$$

where $\mathbb{E}[\cdot]$ is under the probability measure induced by $a = p$, i.e., the agent is working all the time before termination. Note that because the agent always enjoys some non-negative rents, in this problem the agent's participation constraint never binds. Denote the solution to this problem as Π^*.

3 State Variables in Optimal Contracting

We employ a relaxation method in this paper, and solve for the optimal contract by keeping track of two state variables: the agent's continuation payoff, and the agent's marginal utility from consumption. In this section, based on the agent's joint deviation strategy, we first specify the necessary conditions on the evolutions of two state variables. Then Section 4 solves the relaxed problem with these necessary conditions only, and verifies that the obtained solution satisfies the original constraints.

3.1 Continuation Payoff and Incentive-Compatibility Constraint

Given a contract $\Pi = \{\tau, \{c\}, F_\tau\}$, we introduce the agent's continuation payoff, W_t, as,

$$W_t = \mathbb{E}_t\left[\int_t^\tau e^{-r(s-t)}u(c_s)\,ds + e^{-r(\tau-t)}\frac{u(rF_\tau)}{r}\right],$$

which is the agent's future value from the contract Π if the agent keeps working until termination, and conducts no savings. It is important to note that, in equilibrium, W has to be the agent's optimal value among all possible deviation strategies.

In this paper, we use the incentive-compatibility constraint exclusively for the agent's effort choice. In other words, at any time t the contract is incentive-compatible, if the agent's single effort deviation (from the equilibrium effort $a = p$, to $a = 0$ or $a = \overline{p}$), while fixing the follow-up effort-consumption policies, cannot improve the agent's payoff. This clarification is important, because as we will show shortly, the agent's joint deviation involving shirking and saving is key to understanding our results.

Using a martingale representation result, we can write the evolution of the agent's continuation payoff as,

$$dW_t = rW_t dt - u(c_t)\, dt + \beta_t (dN_t - pdt), \tag{5}$$

where the martingale loading β_t measures the responsiveness of the agent's continuation payoff W_t to the unexpected performance $dN_t - pdt$ under the equilibrium working effort.

As standard in this literature (e.g., Sannikov (2006), He (2007a), etc.), β_t controls the agent's incentives to exert effort, *fixing the agent's equilibrium consumption plans as recommended by the contract*. Intuitively, the agent's local effort decision is as follows. Choosing a_t affects the agent's personal cost $b\left(1-\frac{a_t}{p}\right)dt$; however, this also sets the drift of dN_t to be a_t in his continuation payoff. As a result, the agent is solving

$$\max_{a_t \in \{0, p, \overline{p}\}} b\left(1-\frac{a_t}{p}\right) + \beta_t a_t.$$

Because the objective is linear in a, to implement the interior working effort $a = p$, β_t has to equal $\frac{b}{p}$.

In fact, under the framework of binary effort levels, to motivate working against shirking, the incentive β_t must be *no less than* $\frac{b}{p}$. Because the same argument can be applied to the effort choice between "working" and "myopic", to prevent $a = \overline{p}$, β_t must be *no greater than* $\frac{b}{p}$. As a result, $\beta_t = \frac{b}{p}$. In words, because highly powered incentives can induce some myopic actions from the agent, investors never impose excessive incentives on the agent. In contrast, in the standard binary-effort setting without private savings (e.g., Sannikov (2006), He (2007a)), the binding incentive-compatibility constraint is a result of optimal contracting.

For illustration, consider the following discrete-time example. Ignore discounting ($r = 0$), and set $p = 0.5$, $b = 2$. Suppose at date t before consumption, the agent is promised with a continuation payoff of 11. Consider a contract where the agent's date t consumption $c_t = 1$, and assume that $u(1) = 1$. Then his post-consumption continuation payoff at t is 10, and in equilibrium, for promise keeping, we must have

$$0.5 \times W_{t+1}^1 + 0.5 \times W_{t+1}^0 = 10,$$

where W^1_{t+} (W^0_{t+}) is the pre-consumption continuation payoff at date $t+1$ with (without) success along the equilibrium path. This condition is exactly reflected by the drift in (5). Now it is clear that the reward difference $W^1_{t+} - W^0_{t+}$ pins down the agent's working incentives. To implement interior working, however, it must be the case that $W^1_{t+1} = 12$ and $W^0_{t+1} = 8$. If not, say $W^1_{t+1} = 13$ (11) and $W^0_{t+1} = 7$ (9), then the agent will take the myopic (null) action. Here, the incentive loading $\beta_t = W^1_{t+} - W^0_{t+} = \frac{b}{p} = 4$. Note that if the agent shirks, his deviation payoff is $b + W^0_{t+1} = 10$, which is just his date-t post-consumption payoff under working along the equilibrium path.

This binding incentive-compatibility structure has important implications on the wage process studied in the next section, where we consider the agent's joint deviation in both the effort and savings policies. The following proposition summarizes the above discussion.

Proposition 1 *For any employment contract* Π *to be incentive-compatible, the agent's continuation payoff* W *evolves according to (5), and* $\beta_t = \frac{b}{p}$ *for all* $t \in [0, \tau)$. *The agent is indifferent between shirking* ($a = 0$) *and working* ($a = p$).

3.2 Marginal Utility

Now we investigate the agent's saving incentives; for a similar argument, see Kocherlakota (2004). Denote the agent's marginal utility at time t as $M_t = u'(c_t)$. To rule out private savings, the agent's expected marginal utility must be non-increasing. Formally, the marginal utility must follow a supermartingale, i.e.,

$$\mathbb{E}^a_t \left[M_{t+dt} \right] \leq M_t. \tag{6}$$

As a salient feature of any dynamic agency problem, in (6), $\mathbb{E}^a_t [\cdot]$ is the expectation operator under the probability measure induced by the endogenous effort choice a. Necessarily, under the equilibrium working effort, condition (6) requires that

$$(1 - pdt) \cdot M^0_{t+} + pM^1_{t+} dt \leq M_t, \tag{7}$$

where we denote M^0_{t+} (M^1_{t+}) as the marginal utility at $t + dt$ without (with) success.

More importantly, because the agent obtains the same payoff from shirking, the same *supermartingale* results hold for the off-equilibrium shirking effort $a = 0$. Specifically, when the agent shirks—so for sure there is no jump—condition (6) requires that

$$M_{t+}^{0} \leq M_t. \tag{8}$$

In words, this condition is derived from the agent's potential joint deviation. Suppose not; so $M_{t+}^{0} > M_t$, i.e., $c_{t+}^{0} < c_t$ in terms of wages. Then the agent will shirk, and save in the meantime. Shirking gives the agent the exact same equilibrium payoff, and the concurrent saving (when $a = 0$) leads to a strictly positive consumption smoothing gain. Therefore, any incentive-compatible and no-savings contract cannot specify $M_{t+}^{0} > M_t$.

Following the previous discrete-time example, let us assume that $u'(1) = 1$, $u'(0.8) = 1.1$, and $u'(1.2) = 0.9$. The contract can potentially assign the date $t+1$ consumption with (without) success, c_{t+1}^{0} (c_{t+1}^{1}), to be 1.0 (1.2), because they satisfy both conditions in (7) and (8). Now suppose instead the contract reduces the wage after poor performance, i.e., specify c_{t+1}^{0} (c_{t+1}^{1}) to be 0.8 (1.2). Then it is no longer feasible as it violates condition (8), even though the no-savings condition (7) under the equilibrium measure holds in equality. In fact, by deviating from working to shirking, the agent's pre-consumption continuation payoff at t is still 11. Now since the agent can concurrently save 0.1 for date $t+1$ as there is no success for sure tomorrow, (and following the equilibrium strategies from then on), his pre-consumption deviation value at t becomes

$$11 + 2u(0.9) - u(1) - u(0.8) > 11,$$

which suggests that this contract fails to be incentive-compatible and no-savings.

The agent's marginal utility serves as the second state variable in this model,[11] and the above two conditions (7) and (8) constitute the necessary conditions for any contract that implements working while prevents the agent from saving. In the next section, we will verify that in the optimal contract, condition (8) binds always, and we have

$$M_{t+}^{1} \leq M_{t+}^{0} = M_t.$$

[11] At the first sight it seems that we can equivalently set the non-decreasing wage c as the state variable. However, for a potential Pareto improvement, the optimal contract should allow for a simple randomization technique, where the marginal utility becomes the key in preventing the agent's private savings. We formally show the concavity of the investors' value function in Proposition 3, which implies that randomization is suboptimal.

Translating it to a statement of wage c, the agent's wage level remains constant without jumps, but might rise in response to a success.

Additionally, non-decreasing wages imply that even if the agent's poor performance triggers the punishing termination, his consumption level cannot fall.[12] Therefore, as we will show shortly, the optimal contract features a positive severance pay upon the agent's dismissal.

4 Optimal Contracting

We have two state variables in this model: the agent's continuation payoff W, and the agent's marginal utility M. In the previous section, we have specified three necessary conditions for feasible Π: $\beta_t = \frac{b}{p}$ in (5) for incentive-compatibility, and (7) and (8) to rule out private savings. We will solve the relaxed problem using these necessary conditions, and then verify that the solution in fact solves the original problem.

4.1 Construction of Value Function $J(W, M)$

Recall that in our model the agent can generate K cashflows, and in the Appendix we construct the investors' value function $J^K(W, M)$ iteratively. Because the key properties of value function is independent of K, in the main text we take K to infinity, and denote $J(W, M) = J^\infty(W, M)$.

Several functions are useful in later analyses. To express the agent's utility and consumption in terms of the agent's marginal utility $M \in [\underline{\gamma}, \gamma]$, we focus on the strictly concave part of (3),[13] and define the utility function

$$U(M) = u(c) = 1 - \frac{M}{\gamma}, \tag{9}$$

and the consumption function

$$c(M) = \frac{1}{\gamma} \ln \frac{\gamma}{M}.$$

[12] After the termination time τ, there is no cashflow output and the agent's optimal consumption rule implies that $M^0_{t+} = M_{t+} = M_t$ always. Apply the above joint-deviation strategy to the instant right before the termination, one can show that the agent's consumption never falls.

[13] First, $M \geq \gamma = u'(0)$; second, we focus on the case $M < \underline{\gamma}$, as the analysis is trivial when the agent's consumption c reaches the linear part in (3). In this case the agent becomes risk-neutral, and his future marginal utility cannot fall below $\underline{\gamma}$ (otherwise he will conduct saving under the equilibrium working effort).

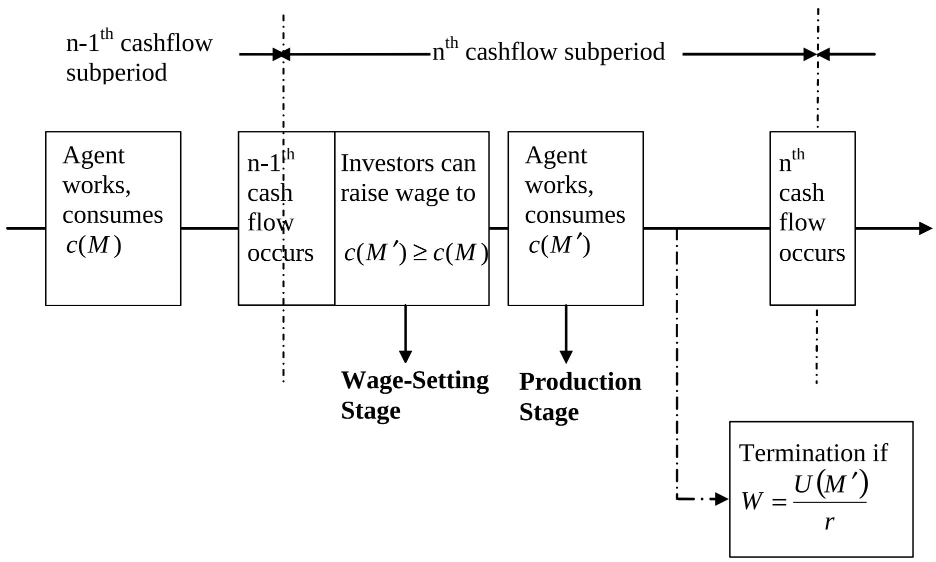

Figure 2: Time-line for the optimal contracting. The n^{th} cashflow subperiod starts with the occurrence of $n-1^{th}$ cashflow. Investors can raise the agent's wage (wage-setting stage) from $c(M)$ to $c(M')$. Afterwards the agent works to produce the n^{th} cash flow (production stage). The project is liquidated and the agent is fired if his continuation payoff W hits $\frac{U(M')}{r}$ first. These two stages repeat themselves for the following subperiods.

The fact that $U(M)$ is linear in M, a unique property of CARA utility, simplifies the following derivation greatly. When the agent is fired, to fulfill the continuation payoff W investors will simply pay the agent $F_\tau = \frac{u^{-1}(rW)}{r}$. Therefore we define the investors' value function at termination as

$$J^L(W) = \frac{u^{-1}(rW)}{r}. \tag{10}$$

For expositional purposes, we spell out the optimal policy first. Figure 2 depicts the time-line where the subperiod of n^{th} cashflow is highlighted, where $n < K$. As shown, we decompose each subperiod into the wage-setting stage and the production stage. Given an occurrence of cashflow, in the wage-setting stage investors have the option to raise the agent's wage to $c(M') \geq c(M)$. Then in the production stage the agent keeps working ($a = p$) until the n^{th} cashflow realizes, or he is fired before the n^{th} cashflow realization.

In the following constructions, we take the value function J as given, and then consider the production stage and wage-setting stage for the previous cashflow period. For detailed iterative constructions, see the Appendix.

4.1.1 Production Stage: Construction of $\widetilde{J}(W, M)$

We start our analysis backward in the production stage. Investors are given the agent's continuation payoff W, his marginal utility M, and the value function $J(\cdot,\cdot)$ in the next subperiod once a cashflow is realized. As shown in Figure 3, J incorporates the investors' option value to raise the agent's wage before they ask the agent to work; therefore we denote the value function in the production stage as $\widetilde{J}(W, M)$ which excludes this option value. We have shown that

$$dW_t = rW_t dt - U(M_t) - bdt + \frac{b}{p} dN_t. \tag{11}$$

As verified in the next section, without success the agent's marginal utility M remains constant, and W evolves as

$$dW_t = rW_t dt - U(M) - bdt; \tag{12}$$

Once a cashflow occurs, $W_{t+dt} = W_t + \frac{b}{p}$, and investors obtain a value $J\left(W_t + \frac{b}{p}, M\right)$.

There is one important point regarding termination. To respect condition (6), given a marginal utility M, any continuation payoff $W < \frac{U(M)}{r}$ is infeasible.[14] In fact, $W - \frac{U(M)}{r}$ reflects the positive rent enjoyed by the agent. When $W = \frac{U(M)}{r}$, which is on the termination curve $l(M) = \frac{U(M)}{r}$ in Figure 3, zero future rent triggers an immediate firing. The implication is that, even if the agent's performance is poor, he is given a severance pay $\frac{c(M)}{r}$ in the "punishing" termination event. Otherwise, the agent will shirk, and save his current wage to smooth his consumption after his dismissal.

The above discussion implies that when $W = \frac{U(M)}{r}$, the firm is liquidated, and

$$\widetilde{J}\left(\frac{U(M)}{r}, M\right) = J^L\left(\frac{U(M)}{r}\right) = -\frac{c(M)}{r}.$$

In the region $W > \frac{U(M)}{r}$, (11) implies that the Hamilton-Jacobi-Bellman (HJB) equation for investors' value function $\widetilde{J}$ satisfies

$$r\widetilde{J}(W, M) = pY - c(M) + p\left[J\left(W + \frac{b}{p}, M\right) - \widetilde{J}(W, M)\right] + \widetilde{J}_W(W, M)\left(rW - U(M) - b\right). \tag{13}$$

[14] This result hinges on the fact that $U(M)$ is weakly convex (in the CARA case it is linear)—see a discussion in Section 5.1. To see this, note that no private saving implies that $M_t \geq \mathbb{E}_t^a(M_s)$ for $s > t$ (s could be larger than τ, in which case the distribution is degenerate). Then according to the definition of W_t, we have $W_t \geq \mathbb{E}_t^a\left[\int_t^\infty e^{-r(s-t)} U(M_s)\, dt\right] = \int_t^\infty e^{-r(s-t)} U(\mathbb{E}_t^a M_s)\, dt \geq \frac{U(M_t)}{r}$, where the first "$\geq$" is due to the possibility of $M_s = \underline{\gamma}$, the second "=" is due to the linearity of $U(\cdot)$ in (9), and the third "$\geq$" is because $U(\cdot)$ is decreasing.

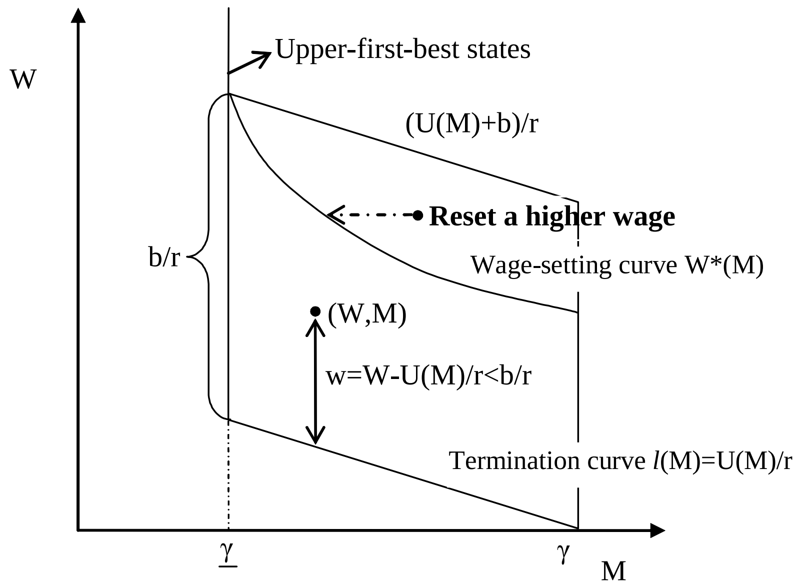

Figure 3: The (W, M) state space. The liquidation line is $l(M) = \frac{U(M)}{r}$, and the wage-setting curve $W^*(M) < \frac{U(M)+b}{r}$. Whenever (W, M) is above the curve $W^*(M)$, the optimal contract resets a higher wage $c(M')$—or a lower marginal utility M'—so that $W = W^*(M')$.

When $W < \frac{b+U(M)}{r}$, a condition to be verified in the next subsection, this ODE admits a closed-form solution:

$$\widetilde{J}(W, M) = [b - rW + U(M)]^{1+\frac{p}{r}} \left[\int_{\frac{U(M)}{r}}^{W} \frac{pY + pJ\left(x + \frac{b}{p}, M\right) - c(M)}{[b - rx + U(M)]^{2+\frac{p}{r}}} dx + J^L\left(\frac{U(M)}{r}\right) b^{-1-\frac{p}{r}} \right]. \tag{14}$$

One can read it as follows: at any state (W', M), investors' instantaneous gain is simply

$$p\left(Y + J\left(W' + \frac{b}{p}, M\right)\right) - c(M),$$

which is the expected value upon success, minus the wage payment. Therefore, the investors' value at state (W, M) is the integration over these instantaneous success gains for $W' < W$, plus the liquidation value without any success $J^L\left(\frac{U(M)}{r}\right)$, all properly discounted based on the Poisson structure.

We list the main properties of the production stage value function $\widetilde{J}$ in the following Proposition. As the fixed-point argument suggests, they are based on the properties of J in the wage-setting stage (listed in Proposition 3) studied in Section 4.1.2.

Proposition 2 *For the production-stage value function $\widetilde{J}$, we have*

1. $\widetilde{J}_W \geq -\frac{1}{\underline{\gamma}}$, *and* $\frac{1}{\gamma r M} < \widetilde{J}_M - \frac{1}{\gamma r}\widetilde{J}_W \leq \frac{1}{\gamma r \underline{\gamma}}$.

2. $\widetilde{J}_{WW} < 0$, $\widetilde{J}_{MM} < 0$, *and* $\widetilde{J}_{WW}\widetilde{J}_{MM} - \left(\widetilde{J}_{WM}\right)^2 > 0$. *Therefore* $\widetilde{J}(W, M)$ *is concave.*

3. $\widetilde{J}_M\left(\frac{b+U(M)}{r}, M\right) < 0$, *and* $\widetilde{J}_{WM} < 0$.

Property one is straight-forward. Because investors can choose to pay (permanently) the agent by at most $\frac{1}{\underline{\gamma}}$ to deliver per unit W, $\widetilde{J}_W$ is bounded by $-\frac{1}{\underline{\gamma}}$. And, as shown in Figure 3, the endogenous termination probability is determined by $w = W - \frac{U(M)}{r} = W - \frac{1}{r} + \frac{M}{\gamma r}$. Therefore, $\widetilde{J}_M - \frac{1}{\gamma r}\widetilde{J}_W$ measures the impact of reducing M (raising the agent's wage) while fixing the termination probability, and the second estimation result follows from the fact that reducing marginal utility has to be permanent.[15]

The second key concavity property implies that any randomization beyond cashflow shocks is suboptimal. As shown in the Appendix, to ensure concavity, the following sufficient condition on the project profitability is required:

$$Y > \max\left(\frac{1}{\gamma r}\left[\frac{\gamma}{\underline{\gamma}} - 1\right]^2, \frac{b}{p\underline{\gamma}}\right). \tag{15}$$

In fact, when $\underline{\gamma} = 0$, the extreme wealth effect dominates the project profitability in some states, and Section 5.4 shows that randomization might be optimal after a long sequence of successes.

Finally, the third property pertains to the optimal wage-setting policy, which will be discussed in the next subsection.

4.1.2 Wage-Setting Stage: Construction of $J(W, M)$

We have obtained investors' value function $\widetilde{J}(W, M)$ in the production stage, where M is taken as given. However, before asking the agent to work, investors have the option to raise the agent's wage (or, reduce M). Specifically, as shown in Figure 3, at the wage-setting stage investors should choose $M' < M$ if $\widetilde{J}(W, M') > \widetilde{J}(W, M)$. Let the optimal marginal utility level M^* given W be

$$M^*(W) = \arg\max_{M' \in [\underline{\gamma}, \gamma]} \widetilde{J}(W, M'), \tag{16}$$

[15]Intuitively, since the future marginal utility $M_s \leq M$, the marginal cost brought on by permanently reducing one unit of M is a weighted average of $-\frac{c'(M_s)}{r}$ in the future, which must belong to $\left(\frac{1}{\gamma r M}, \frac{1}{\gamma r \underline{\gamma}}\right]$.

and define the investors' value function at the wage-setting stage as

$$J(W,M)=\begin{cases}\widetilde{J}(W,M) & \text{if } M\leq M^{*}(W)\\ \widetilde{J}(W,M^{*}(W)) & \text{otherwise}\end{cases}. \tag{17}$$

Simply put, investors reduce M to $M^{*}(W)$ by exercising the option of raising the agent's wage, and the resulting value function J has $J_M \geq 0$ always. This transformation preserves the concavity of J.

The economic rationale behind the wage-setting policy is as follows. From the cost side, as $w = W - \frac{U(M)}{r}$ captures the termination probability (see Figure 3), a lower M reduces w, thereby making the costly termination more likely. On the other hand, due to the agent's risk-aversion, raising wage gives a consumption-smoothing benefit (as the agent's equilibrium consumption pattern is back-loaded), especially when the agent's continuation payoff W is sufficiently high. Consequently, the optimal wage-setting policy equates the marginal cost (brought on by inefficient terminations) with the marginal benefit (for consumption smoothing).

Now we discuss the property 3 in Proposition 2. First, $-\widetilde{J}_M$ captures the marginal benefit from raising wage. Therefore, $\widetilde{J}_{WM} < 0$ implies that for $W > W'$, $-\widetilde{J}_M(W,M) > -\widetilde{J}_M(W',M)$. Intuitively, because a higher continuation payoff gives rise to a larger consumption-smoothing benefit, the marginal benefit of raising wage is greater when W is higher. Second, $\widetilde{J}_M\left(\frac{b+U(M)}{r},M\right) < 0$ implies that the curve $M^{*}(W)$ stays below the line $W = \frac{b+U(M)}{r}$. Put differently, on this line, it is always optimal to set a higher wage. In fact, one can check that, the marginal impact of future termination cost by setting $M^{*}(W)$ slightly below $\frac{b+U(M)}{r}$ is zero (see the Appendix). On the other hand, because $\frac{b+U(M)}{r}$ is the upper bound of the agent's continuation payoff W given M (or $c(M)$), the wage level has to increase once a jump occurs,[16] and the marginal benefit of consumption smoothing is strictly positive. Therefore, raising wages before W reaching $\frac{b+U(M)}{r}$ is optimal. Consistent with this intuition, when the agent becomes risk-neutral at $M = \underline{\gamma}$ so that the consumption-smoothing benefit turns zero, we have $W^{*}(\underline{\gamma}) = \frac{U(\underline{\gamma})+b}{r}$.[17]

We know that $M^{*}(W)$ satisfies $\widetilde{J}_M(W,M^{*}(W)) = 0$ for $M^{*}(W) \in (\underline{\gamma},\gamma)$ in Figure 3. Then, the

[16] $\frac{b+U(M)}{r}$ is equal to the agent's guaranteed annuity utility from consumption, plus his permanent shirking benefit—which is the highest value that investors can possibly deliver given the wage level $c(M)$.

[17] To see this, according to the property 1 in Proposition 2, we have $\widetilde{J}_M = \frac{1}{\gamma r}\left(\widetilde{J}_W + \frac{1}{\underline{\gamma}}\right) \geq 0$ always. Therefore when $W = \frac{U(\underline{\gamma})+b}{r}$, the first-best result holds, $\widetilde{J}_W = -\frac{1}{\underline{\gamma}}$, so $\widetilde{J}_M = 0$.

wage-setting curve $M^*(W)$ is downward sloping, i.e.,

$$M^{*\prime}(W) = -\frac{\widetilde{J}_{WM}}{\widetilde{J}_{MM}} < 0.$$

Therefore we define the inverse function $W^*(M)$, which is the highest continuation payoff given M such that $\widetilde{J}_M$ remains nonnegative.

For states below $M^*(W)$, due to the construction in (17), we still have $J_{WM} = \widetilde{J}_{WM} < 0$; and as in Figure 2, all subperiods start from some state (W_0, M) below $M^*(W)$ so that $J_M(W_0, M) \geq 0$. Then, along the equilibrium path without success, investors will not exercise the option of raising wage. To see this, the marginal benefit of raising wage is smaller for subsequent lower continuation payoffs W_t given no success, as $-J_M(W_t, M) < -J_M(W_0, M)$ for $W_t < W_0$ due to $J_{WM} < 0$. Therefore, if it is optimal not to raise wage at W_0 initially, then it must be optimal to keep the same wage along the path without success. This result verifies that, we can take M as constant during the production stage in Section 4.1.1.

The following proposition gives properties of the value function $J(W, M)$ based on Proposition 2.

Proposition 3 *The value function $J(W, M)$ satisfies the following properties:*

1. $J_W \geq -\frac{1}{\underline{\gamma}}$, and $\frac{1}{\gamma r M} < J_M - \frac{1}{\gamma r} J_W \leq \frac{1}{\gamma r \underline{\gamma}}$.

2. $J_{WW} < 0$, $J_{MM} \leq 0$, and $J_{WW} J_{MM} - (J_{WM})^2 \geq 0$. Therefore $J(W, M)$ is concave.

3. $J_{WM} \leq 0$; $J_M \geq 0$ and $J_M\left(\frac{b+U(M)}{r}, M\right) = 0$.

The above analysis does not cover the upper-first-best region (see Figure 3). There, the (loaded) risk-neutral agent with $W \geq \frac{b+U(\underline{\gamma})}{r}$ consumes wages no lower than $c(\underline{\gamma})$, and obtains $\frac{b}{p\underline{\gamma}}$ from each cashflow realization Y. There is no future costly termination, and the first-best result is achieved. For the analytical form of J in this upper-first-best region, see the Appendix.

4.2 Optimal Contract and Comparison to the Case with Observable Savings

We summarize the optimal contract as follows. The agent is promised with a life-time wage level. If the agent's performance is sufficiently good, he will receive pay raises (as promotions), and these raises are

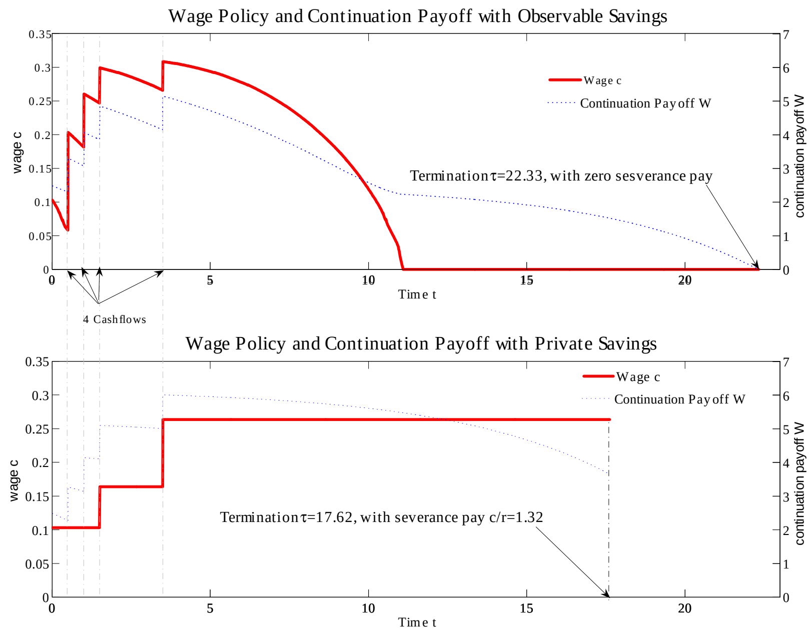

Figure 4: Optimal wage policies and associated continuation payoff evolutions for the cases with private savings (the bottom panel) and the case with observable savings (the top panel). The solid line is for wage process, and the dotted line is for the agent's continuation payoff W. The history consists of 4 cashflows at $t = 0.5$, 1.0, 1.5, and 3.5; and no cashflow afterwards. Parameters are $b = 0.5$, $Y = 20$, $r = 0.2$, $p = 0.5$, $\gamma = 5$, and $\underline{\gamma} = 1$. In the top panel the agent is fired when $W = 0$. In contrast, in the bottom panel the agent is fired when $W = 3.66 = \frac{\overline{U}(M)}{r}$, where $M = 1.34$ and $c(M) = 0.26$.

permanent. On the other hand, given poor performance, the agent is dismissed with a severance pay to support his post-firing consumption at his current wage level, and he loses potential future pay raises.

The possibility of private savings has a dramatic impact on the optimal wage policy. Keep the same model, but consider the only modification that the agent's savings are (publicly) observable. In this case, the agent's consumption, which is just the wage paid by investors, is contractible. As a dynamic agency problem with hidden actions studied in Sannikov (2006), the agent's continuation payoff W is the only state variable in solving for the optimal contract (see the Appendix), and the optimal wage becomes a function of the continuation payoff W.[18]

We graph the optimal wage policies (left scale) and associated continuation payoff dynamics (right scale) in Figure 4. The history consists of 4 cashflows occurred in $t = 0.5$, 1.0, 1.5, and 3.5; afterwards the agent experiences no successes even with his effort input. The top (bottom) panel is for the case with

[18] In Sannikov (2006), there are no myopic actions. But because in the optimal contract the agent's incentive-compatibility constraint is binding, for observable savings the restriction brought on by myopic actions is redundant.

observable (private) savings; for better comparison, we use the same scale for both cases.

In the top panel with observable savings, the agent's wages exhibit a quite sensitive response (a zig-zag pattern) to his performance, and wages drop given no success. In contrast, in the bottom panel with private savings, the response is muted: wages are downward rigid, and pay raises are less frequent (only twice given four cashflows).[19] One take-away is that, because the agent's potential deviation strategies (e.g., shirking and saving) make certain sensitive wage policies non-viable, compensation contracts with low wage-performance sensitivities become optimal.

Finally, the wage policy also has noticeable impact on the termination policy. Given the long poor performance after $t = 3.5$ in Figure 4, in the top panel with observable savings, the agent's continuation payoff falls at a lower rate than in the bottom panel; this is due to the downward wage adjustment in the top panel. As a result, the life span of the firm is longer (22.23 versus 17.62) when the agent cannot privately save. Besides, in our model, the agent walks away with a positive severance pay, while he will receive no severance pay if savings are observable.

The seemingly inefficient compensation patterns in the bottom panel, i.e., low compensation-performance sensitivities and generous severance payments even after poor performance, are usually viewed as symptoms of malfunctioned corporate governance (e.g., Bebchuk and Fried (2004)). However, this paper shows that under realistic contracting friction assumptions, they are actually part of the optimal contract. By providing a concrete example, this paper raises a critique to the Bebchuk and Fried's logic flow from the observed "inefficient" forms of executive compensation to the failed corporate governance (for a similar point, see Core, Guay and Thomas (2004)).

4.3 Justification of the Optimal Contract

4.3.1 Verifying the Optimal Contract

Introduce the investors' auxiliary gain process G_t as

$$G_t = -\int_0^t e^{-rs} c_s ds + \int_0^t e^{-rs} Y dN_s + e^{-rt} J\left(W, M\right). \tag{18}$$

[19]However, the corresponding levels of continuation payoff W are indeed important in comparing the wage responsiveness for each particular event. For instance, for the 4^{th} cashflow realization at $t = 3.5$, the observable-savings case features a smaller pay raise than the private-savings case; the reason is simply because the former has a lower continuation payoff (so each dollar is more valuable) than the latter.

As discussed in Section 3, for any employment contract Π to be incentive-compatible and no-savings, we have $dW = (rW - U(M) - b)\,dt + \frac{b}{p}dN_t$, and combining (7) and (8) gives (recall that M^1_{t+} is the marginal utility at time $t + dt$ if a jump occurs, and M^0_{t+} if not):

$$M^0_{t+} \leq M, \text{ and } (1 - pdt) \cdot M^0_{t+} + pM^1_{t+}dt \leq M. \tag{19}$$

Note that the only relevant control is the evolution of the agent's marginal utility M, as dW is dictated by the binding incentive-compatibility constraint.

The investors' expected instantaneous gain can be written as

$$\begin{aligned}\mathbb{E}_t\left[e^{rt}dG_t\right] &= \left[-rJ - c(M) + p\left(Y + \left[J\left(W + \frac{b}{p}, M^1_{t+}\right) - J(W, M)\right]\right) + J_W \cdot (rW - U(M) - b)\right]dt \\ &\quad + \left[J\left(W, M^0_{t+}\right) - J(W, M)\right].\end{aligned}$$

Given the fact that $J_M \geq 0$ and $J_{MW} < 0$, i.e., there is more wage raising benefit when the agent's continuation payoff is higher (recall the discussions in Section 4.1.2), we can show that the optimal policy is $M^0_{t+} = M^1_{t+} = M$.[20] Intuitively, as in typical optimal contracting problems with moral hazard, investors want to raise (cut) the agent's wage with (without) a success; but as discussed in Section 3.2, it is infeasible for incentive-compatible and no-savings contracts. This result suggests that the wage remains constant without a jump; and keep in mind that, once M^1_{t+} exceeds $M^*\left(W + \frac{b}{p}\right)$, an immediate pay raise will occur after a success.

Due to the construction in Section 4.1, we have $\mathbb{E}_t\left[e^{rt}dG_t\right] = 0$ under the optimal policy, and it is non-positive for other incentive-compatible and no-savings contracts. Then a standard verification argument leads to the following theorem.

Theorem 1 *Consider the stationary case $K \to \infty$. Take the value function $J(W, M)$ and the wage-setting curve $M^*(W)$ as given. Under the optimal contract Π^*, we have*

$$dW_t = (rW_t - U(M_t) - b)\,dt + \frac{b}{p}dN_t,$$

[20]Remember that there is a wage raise whenever $M_t > M^*\left(W^1_{t+}\right) = M^*\left(W_t + \frac{b}{p}\right)$. We maximize $pJ\left(W + \frac{b}{p}, M^1_{t+}\right)dt + J\left(W, M^0_{t+}\right)$ subject to (19). First, as $J_M \geq 0$, without loss of generality we can set $M^0_{t+} = \frac{M - pM^1_{t+}dt}{1-pdt}$; then $M^0_{t+} \leq M$ implies that $M^1_{t+} \geq M$. Given these conditions, a simple calculation shows that $M^0_{t+} = M^1_{t+} = M$ is optimal. Also, the concavity $J_{MM} < 0$ implies that the first-order condition is sufficient.

and

$$M_{t+dt} = M_t \cdot \mathbf{1}_{\{dN_t=0\}} + \min\left(M^*\left(W_{t+dt}\right), M_t\right) \cdot \mathbf{1}_{\{dN_t=1\}}.$$

The employment is terminated whenever $W_\tau = \frac{U(M_\tau)}{r}$*, and the agent gets a severance pay* $F_\tau = \frac{c(M_\tau)}{r}$*. When* $W_t > \frac{U(\underline{\gamma})+b}{r}$*,* $M_t = M^*(W_t) = \underline{\gamma}$*, and the first-best result is achieved: investors pay the agent* $\frac{1}{\underline{\gamma}}\left[W_t - \frac{U(\underline{\gamma})+b}{r}\right]$*, ask him to work forever, and pay him* $\frac{b}{\underline{\gamma}p}$ *whenever a cashflow occurs.*

Finally, recall that by considering only the necessary (local) conditions for Π to be incentive-compatible and no-savings, we are solving a relaxed version of the investors' problem. However, one can easily verify that, given the above non-decreasing wage contract with a severance payment described in Theorem 1, the agent's optimal strategy is to exert working effort and maintain zero savings always. Therefore, the resulting solution satisfies the original restrictions in the investors's problem, and the contract is indeed optimal.

4.3.2 Optimality of Implementing the Normal Working Effort

Suboptimality of the Null (Shirking) Effort When the null effort is implemented, there is no jump in the next instant, and to prevent the agent from saving M_t must be non-increasing. This is the same wage policy as in implementing the working effort $a = p$. For shirking to be dominated, we must have a positive net gain of a jump, i.e.,

$$Y + J\left(W + \frac{b}{p}, M\right) - J(W, M) \geq 0 \text{ for all } (W, M).$$

Because J is concave in W, and linear in W in the upper-first-best region, the necessary and sufficient condition is,

$$Y \geq \frac{b}{p\underline{\gamma}} \Leftrightarrow Y \geq J(W, M) - J\left(W + \frac{b}{p}, M\right) \geq -J_W(W, M)\frac{b}{p}.$$

This is just the standard condition for the suboptimality of shirking when the agent becomes risk-neutral in the upper-first-best states. Intuitively, for working to be optimal, the expected cashflow pY should be higher than the upper-bound of the agent's equivalent "monetary" effort cost, which is $b/\underline{\gamma}$ when the agent becomes sufficiently wealthy.

Suboptimality of the Myopic Action If the myopic effort $a = \overline{p}$ is implemented, then for some $\beta_t \geq \frac{b}{p}$, we can write the evolution of W_t as,

$$dW_t = rW_t dt - U(M)\,dt + \frac{b}{p}(\overline{p} - p)\,dt + \beta_t\left(dN_t(\overline{p}) - \overline{p}dt\right).$$

We need to show that G in (18) has a negative drift once $\overline{p}$ is implemented, subject to the incentive-compatibility and no-savings conditions.

Let us pause to discuss the economic intuition. There is a non-contractible loss Δ due to the myopic action, which hurts the firm's value along other dimensions. On the benefit side, the myopic action boosts the cashflow intensity to $\overline{p} = p + \epsilon$. Are there any other benefits by implementing the myopic action in this model?

The answer is yes. Recall that in Section 3.2, the binding incentive-compatibility constraint $\beta_t = \frac{b}{p}$ plays a key role. Once setting $\beta_t > \frac{b}{p}$, the agent's incentive-compatibility constraint is slack, and the key condition (8) no longer holds. In other words, under a highly-powered incentive scheme, the optimal contract punishes shirking severely which in turn deters the agent's joint-deviation strategy of "shirking and saving." As a result, resetting a lower wage level without success—a potentially value-improving policy—becomes possible.

In this case, because the unidimensional variable M is no longer sufficient to capture the agent's private-saving incentive (which depends on the entire continuation contract), it is difficult to pinpoint the exact magnitude of the benefit by adjusting the wage downward. Fortunately, we can utilize the *necessary* no-savings condition (6) *under the equilibrium effort choice* $a = \overline{p}$ to bound this benefit, which says that

$$M \geq M^1_{t+}\overline{p}dt + M^0_{t+}\left(1 - \overline{p}dt\right). \tag{20}$$

Essentially, this condition places a bound on $M^0_{t+} - M$, which is the increment of M (or, reduction of wage) without success. Based on (20), in the Appendix we derive a condition for Δ to offset this upper-bound contractual benefit. Because the actual benefit (subject to additional constraints with respect to the agent's other deviating strategies) must be lower, we essentially provide a sufficient condition for the suboptimality of implementing the myopic action.

5 Generalizations and Extensions

5.1 General Utility Functions

The adoption of CARA utility is only for exposition purposes. This section extends our analyses to a general utility function $u(\cdot)$ that satisfies the bounded-below marginal utility in condition (2). Similar to (9), by writing $g(c) = u'(c)$, we define the agent's utility, as a function of the marginal utility M, to be $U(M) = u\left(g^{-1}(M)\right)$. Now the termination boundary, denoted as $l(M) = \frac{U(M)}{r}$, is no longer a line as in the CARA case (see Figure 3). For the concavity of the value function J, we require that the domain $\{(W, M) : W \geq l(M)\}$ to remain convex—in other words, $l(M)$ is a convex function. One can easily check that $l(M)$ is convex if and only if $u''' > \frac{(u'')^2}{u'}$.[21]

The structure of the resulting optimal contract remains unchanged: wages $\{c\}$ are nondecreasing; the agent works for potential pay raises; and the agent's poor performance leads to dismissal, but he walks away with a severance payment $\frac{c_\tau}{r}$. Interested readers can find detailed constructions in the Appendix, including the variations of Proposition 2 and Proposition 3.

5.2 Renegotiation-Proof Contract

If two parties in this model can renegotiate whenever the original contract can be Pareto improved, the value function $J(W, M)$ must have a non-increasing slope with respect to W.[22] For simplicity, in this section we assume that the liquidation value $L > 0$ is relatively large. Put differently, we consider a moderate impact of renegotiation by focusing on the case where the termination inefficiency is not excessive.

Similar to Section 4.1, we can construct the value function $J^{RP}(W, M)$ recursively (see Appendix for details). Analogous to the unidimensional result in DeMarzo and Sannikov (2006), $J^{RP}(W, M)$ features a renegotiation boundary $\underline{W}(M)$, which is the lower bound of the agent's continuation payoff W during the equilibrium employment path at the wage level $c(M)$, and $J_W(\underline{W}(M), M) = 0$. When the liquidation value L is relatively large, the renegotiation curve $\underline{W}(M)$ (which might bind at $\frac{U(M)}{r}$) is strictly below

[21] The class of CRRA (Constant Relative Risk Aversion, or power) utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ satisfies this condition. Again this property is for convenience only; one can still construct the value function if $l(M)$ is concave, and some randomization will be called upon for the states where $W = l(M) - \varepsilon$.

[22] The definition of renegotiation-proofness here is the same as in DeMarzo and Fishman (2006) and DeMarzo and Sannikov (2007), which is equivalent to the contract being sequentially undominated (in terms of parties payoffs); see Hart and Tirole (1988). In contrast, Hart and Moore (1998) use a different approach. See related comments in DeMarzo and Fishman (2006).

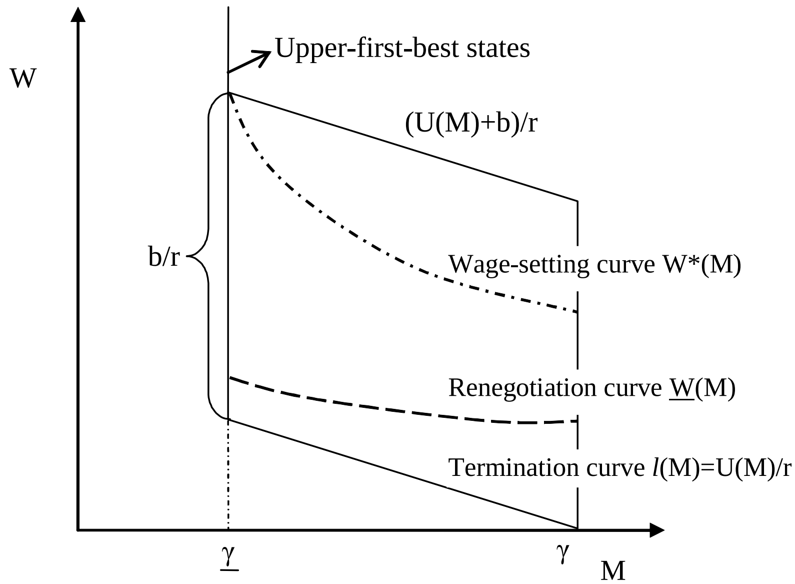

Figure 5: The renegotiation-proof contracts. There exists a renegotiation curve $\underline{W}(M) > \frac{U(M)}{r}$, such that $J_W(\underline{W}(M), M) = 0$, $J_W(W, M) < 0$ for $W > \underline{W}(M)$. Both parties run a lottery once $W = \underline{W}(M)$: with probability $\frac{b+U(M)-r\underline{W}(M)}{\underline{W}(M)-\frac{U(M)}{r}}dt$ the firm is liquidated; otherwise, the agent stays at the same point. When the liquidation value L is relatively large, The wage-setting curve $W^*(M)$ is above the renegotiation curve $\underline{W}(M)$.

the wage-setting curve $W^*(M)$ (see Figure 5).

The renegotiation-proof contract is similar to the result in Section 4.2. However, when poor performance drives W down to $\underline{W}(M)$, both parties run a lottery. The agent is fired with probability

$$\frac{b + U(M) - r\underline{W}(M)}{\underline{W}(M) - \frac{U(M)}{r}}dt;$$

otherwise, the agent stays at $\underline{W}(M)$. Under this lottery, at $W = \underline{W}(M)$, without success the agent's (expected) dW remains $[r\underline{W}(M) - U(M) - b]\,dt$ as in (12). For detailed constructions, see the Appendix.

5.3 The Complete Contract with Multi-Tasking: A Convergence Result

By treating the myopic action loss Δ as non-contractible, our contracting space is somewhat *incomplete*. How far is our optimal contract away from the optimal *complete* contract? Can the non-contractibility be justified by a fixed information acquisition cost in the complete contract paradigm?

To address these issues, in the spirit of Holmstrom and Milgrom (1991), we embed a multi-tasking problem into the main model. Assume that the firm's operation involves another business activity, which brings investors an instantaneous *value increment* as,

$$dQ_t = \mu dt + \sigma dZ_t,$$

where $\{Z_t\}$ is a Brownian process independent of $\{N_t\}$. Here dQ_t, as the agent's second soft performance measure, is observable and contractible; it can also be interpreted as the (noisy) change of the firm's long-run value. To capture the "softness" of the dQ measure, in the following analysis we consider the case where σ is sufficiently large.

Neither shirking or working has any impact on the drift μ in dQ_t. However, once the agent takes the myopic action, μ drops by Δ as the agent transfers his effort allocation from the soft performance dQ_t to the hard performance dN_t. For simplicity we take $\mu = 0$; that is, if the contract chooses to ignore dQ_t, then the optimal (incomplete) contract is just the one derived in Section 4.

In contrast to the case where Δ is non-contractible, now investors can raise the incentive loading on dN_t, while using dQ to prevent the agent from taking myopic actions. To be concrete, the contract can specify the evolution of the agent's continuation payoff as

$$dW_t = (rW_t - U(M_t))\, dt + \beta_t (dN_t - pdt) + x_t dQ_t, \tag{21}$$

where the incentive loading $\beta_t = \frac{b}{p} + k_t > \frac{b}{p}$. Now if

$$x_t \geq \frac{k_t(\overline{p} - p)}{\Delta} = \frac{k_t \epsilon}{\Delta},$$

then the agent will be refrained from the myopic action: by taking $a = \overline{p}$, the agent gains $k_t\epsilon$ from dN_t, but this gain is offset by the loss $x_t\Delta$ from dQ_t.

As discussed in Section 4.3.2, $\beta_t > \frac{b}{p}$ leads to a benefit in relaxing the no-savings constraint. Because the agent will incur a penalty $pk_t dt$ by shirking, which could dominate the consumption smoothing gain, now investors may specify an increase of M^0_{t+} on the path without success. However, on the cost side, since dQ is noisy, the unnecessary punishment—which makes the costly termination more likely—lowers the investors' value. In addition, the extra noise leads to certain inefficiencies due to the agent's risk-aversion. Therefore, in (21), imposing loading $x_t > 0$ on the agent's continuation payoff is costly.

Not surprisingly, in the Appendix we show that when $\sigma \to \infty$, the cost mentioned above becomes overwhelmingly large. This mandates a dying x_t, and as a result, k_t converges to zero.[23] In the limiting case, the shirking penalty $pk_t dt$ is negligible, and the no-savings constraint implies that $M_{t+}^0 - M_t$ goes to zero. As a result, the value from the complete contract converges to the one from the incomplete contract derived in Section 4. Intuitively, when the information precision of the soft performance goes to zero, the contract should simply ignore such extremely noisy signals, just as the incomplete contract does. Therefore, if there is some positive transaction cost in procuring the extremely "soft" information dQ, then the "incomplete" contract derived in Section 4 is optimal even in the paradigm of complete contracts.

5.4 Extreme Wealth Effect when $\underline{\gamma} = 0$

We derived the optimal contract under the technical condition (2) which requires the agent's marginal utility to stay above $\underline{\gamma} > 0$. As discussed after (2) in Section 2.2, this condition places an upper-bound on the effective monetary effort cost, and the wealth effect is only "moderate." The direct consequence is that in our model, the first-best results can be reached given a long sequence of successes. In contrast, the literature has found that the extreme wealth effect leads to the agent's retirement with a high perpetuity compensation after a sufficiently good history (e.g., Sannikov (2006), Spear and Wang (2005)).

To study the case of $\underline{\gamma} = 0$ with the extreme wealth effect, we investigate the optimal contract for $u(c) = 1 - e^{-\gamma c}$, where we remove the upper-linear part in (3). In this case, as M goes to zero, the agent's effective monetary effort cost $\frac{b}{M}$ explodes. Therefore, after a long sequence of successes, shirking becomes optimal when the agent's consumption is sufficiently high. As discussed in Section 4.3.2, shirking should be implemented whenever the total value given a jump is below the current value, i.e.,

$$Y + J\left(W + \frac{b}{p}, M_{t+}^1\right) < J(W, M).$$

When M is sufficiently small, the above condition will hold for some $W > \frac{U(M)+b}{r} - \frac{b}{p}$. To see this, after a jump, $W + \frac{b}{p} > \frac{U(M)+b}{r}$ which is the upper bound of W given M (see footnote 16). Then given a high

[23] When interpreting dQ_t as long-run performances, we are not saying that the agent should receive zero long-term incentive packages. In fact, by introducing another primary long-term value process P, our model can be further extended to the case where the agent has a fixed long-term incentive package regarding P to align the agent's interest with investors. Put differently, in the analysis here, we are focusing on those extra soft long-run performances dQ that are costly to "contract on," just as in Holmstrom and Milgrom (1991).

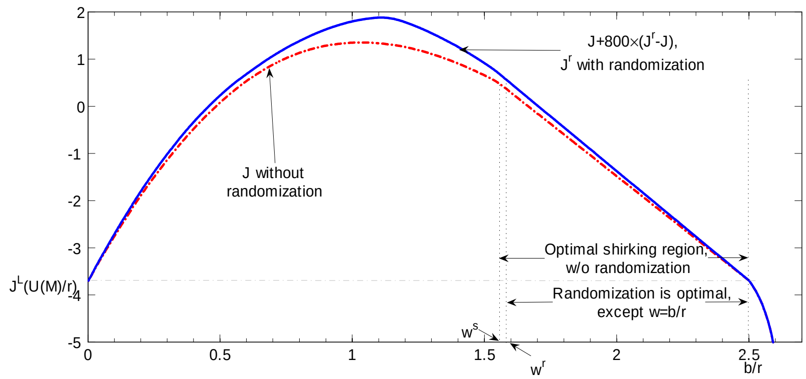

Figure 6: Randomization becomes optimal for a sufficiently small M when the wealth effect is extreme ($\underline{\gamma} = 0$). The parameters are $b = 0.5$, $Y = 5$, $r = 0.2$, $p = 0.5$, $\gamma = 5$, and we plot two value functions with $M = 0.124$. The value function without randomization J is shown in the dash-dotted line, and shirking is invoked once $W > \frac{U(M)}{r} + w^s$. For better illustration, we plot $J^r + 800 \times (J^r - J)$ in the solid line, where J^r is the value function with randomization. In the optimal contract, a randomization is called upon when $W > W^r = \frac{U(M)}{r} + w^r$.

current wage for the agent, a jump must lead to a huge wage increase, and the incremental wage perpetuity could easily outweigh the cashflow profit Y.

Interestingly, whenever shirking is implemented, the two-dimensional value function J becomes non-concave. This calls for an optimal "randomization," which makes the analysis non-tractable. To see the intuition for the non-concavity, let us fix a sufficiently small M (so the wage is sufficiently high). The above argument suggests that for $W < \frac{U(M)+b}{r}$, there will be a threshold such that it is optimal to start implementing $a = 0$. As shown in Figure 6, denote the threshold as W^s (in the figure it is shown as $w^s = W^s - \frac{U(M)}{r}$). One can show that it is optimal to implement shirking over the interval $\left[W^s, \frac{U(M)+b}{r}\right]$, and the HJB equation is

$$rJ = -c(M) + J_W \cdot (rW - U(M) - b).$$

Simple algebra shows that J is linear in W; intuitively, by implementing $a = 0$, the contract executes a linear transition from state $W \in \left[W^s, \frac{U(M)+b}{r}\right]$ to W^s, while keeping M fixed.[24]

An important result ensues here. Because J is linear in W for $W \in \left[W^s, \frac{U(M)+b}{r}\right]$, a simple randomization involving two end points—but with the same M—delivers the same value as $J(W, M)$. Therefore,

[24]Therefore $J(W, M)$ is a linear combination of $J(W^s, M)$ and $J\left(\frac{U(M)+b}{r}, M\right) = -\frac{c(M)}{r}$, which is achieved by asking the agent to consume a perpetuity wage $c(M)$, and shirk forever inside the firm. Also, we can show that $W^s + \frac{b}{p} > \frac{U(M)+b}{r}$; intuitively, working is non-profitable only if a higher wage level is invoked after a jump.

if J_M's are different for these two end points, then a randomization—now involving different M's—can lead to a strict improvement. In fact, as Proposition 3 indicates, one can check that $J_{WM} < 0$ generally holds even with the interim shirking policy. Therefore, $J_{WW} = 0$ but $J_{WM} < 0$ implies that J fails to be concave, and a randomization improves the investors' value. Based on numerical results, Figure 6 plots the value function J without randomization, and the value function J^r after employing an optimal randomization scheme. We also indicate the threshold of shirking (W^s) and randomization threshold (W^r) in the figure.

As $J_{WM} < 0$ suggests, we find that the optimal randomization scheme for (W, M) involves $\left(\frac{U(M_u)+b}{r}, M_u\right)$ and (W_d, M_d) where $M_u < M_d$. In words, when W is sufficiently high that working is suboptimal, the optimal contract will ask the agent to take a lottery. If the agent wins, he is awarded with a higher level of wage perpetuity, and shirks forever inside the firm—the agent is retired with a golden parachute, but still serves as an honored council member to enjoy the private benefit. If the agent loses, then both his wage level and continuation payoff are cut, and as a blue-collar worker he starts working again.

5.5 What if the Agent's Initial Wealth Is Private Information?

Finally we consider a potential adverse selection problem when initiating the contract. If the agent's initial wealth $S_0 > 0$ is his private information, what will be the optimal menu of contracts? Should each contract in the optimal menu take the form we derived in Section 4? Though beyond the scope of this paper, our results do shed some light on these questions.

Suppose that the distribution of S_0 follows a density function $f(\cdot)$, with a bounded support $\left[0, \overline{S}\right]$. Assume that it is optimal to elicit employment (and working) from all agents. One can show that given the contract Π^* derived in Section 4, the agent with savings $S_0 > 0$ has an optimal deviation value as

$$V(M, W, S_0) = W - \Phi(M, 0) + \Phi(M, S_0), \tag{22}$$

where $\Phi(M, S_0) = \frac{1-e^{-\gamma(c(M)+rS_0)}}{r}$. In words, the agent shirks always to get the shirking benefit $W - \Phi(M, 0)$, and eats $c(M) + rS_0$ forever to obtain $\Phi(M, S_0)$ from consumption. The reason is just the wealth effect: a wealthier agent requires higher working incentives, so an agent with positive savings will shirk always under the contract designed for a moneyless agent.

Now we consider the screening problem. Suppose momentarily that investors can only select contract from the class $\{\Pi^*\}$ derived in Section 4. Due to the revelation principle, the optimal menu is a two-dimensional schedule of $(W(S_0), M(S_0))$, such that all agents truthfully report their types. Given the deviation value specified in (22), investors solve the following problem $\mathbb{P}$:

$$
\begin{aligned}
\max_{W(S_0),M(S_0)} \quad & \int_0^{\overline{S}} J(W(S_0), M(S_0)) f(S_0) \, dS_0 \\
s.t. \quad & W(S_0) \geq W\left(\widehat{S}_0\right) - \Phi\left(M\left(\widehat{S}_0\right), 0\right) + \Phi\left(M\left(\widehat{S}_0\right), S_0 - \widehat{S}_0\right) \text{ for all } \widehat{S}_0 < S_0, \qquad (23)
\end{aligned}
$$

where the condition (23) states that the type-S agent finds under-reporting suboptimal.

Because $\Phi_{S_0}(M, 0) = M$, the local incentive-compatibility constraint implies that (the inequality might hold strictly as the agent cannot overreport)

$$
W'_{-}(S_0) = \lim_{\varepsilon \to 0^+} \frac{W(S_0) - W(S_0 - \varepsilon)}{\varepsilon} \geq M(S). \qquad (24)
$$

In fact, (24), which rules out the deviation strategy of under-reporting and consuming right away, should hold for all potential candidate contracts. Now one can first solve the relaxed problem by replacing the global incentive-compatibility condition (23) with the local one (24). Once the solution is obtained, one can check whether the solution satisfies (23), and a sufficient condition is simply $M'(S_0) + \gamma r M(S_0) \geq 0$ for all S_0.[25] If it is indeed the case, then we obtain a solution to the problem $\mathbb{P}$.

In fact, in this case, the derived menu of contracts is truly optimal, *even if investors can design contracts outside the class* $\{\Pi^*\}$. The reason is as follows. Given any state (W, M), without screening considerations, the continuation contract $\Pi^*(W, M)$ is optimal. Therefore, essentially $\Pi^*(W, M)$ reoptimizes the contract at state (W, M)—or, reoptimizes for each type S_0. Therefore, by considering contracts in the form of Π^* and imposing (24) only, we are solving a truly relaxed problem; and if (23) holds automatically, the obtained contract menu is indeed optimal.

However, if (23) does not hold, the problem becomes complicated. First, the solution to problem $\mathbb{P}$ could be involved. More importantly, the solution to $\mathbb{P}$ need not be the solution to the original problem,

[25] Assume that $M(\cdot)$ is differentiable. Given the deviation value $V\left(S_0, \widehat{S}_0\right) = W\left(\widehat{S}_0\right) - \Phi\left(M\left(\widehat{S}_0\right), 0\right) + \Phi\left(M\left(\widehat{S}_0\right), S_0 - \widehat{S}_0\right)$, the sufficient condition for (23) is $\frac{\partial V}{\partial \widehat{S}_0} = V_{\widehat{S}_0} \geq 0$ always. But one can easily show that $V_{\widehat{S}_0} \geq \left(1 - e^{-\gamma r\left(S_0 - \widehat{S}_0\right)}\right)\left(M\left(\widehat{S}_0\right) + \frac{M'\left(\widehat{S}_0\right)}{\gamma r}\right)$.

as investors are free to choose contracts that are more complicated than $\{\Pi^*\}$. One conjecture is that, investors might find that raising the agent's wage even without success—a feature absent from Π^*—is optimal, because this reduces the agent's deviation gain from underreporting and saving. If it is true, then this interesting result shows that, not only is it optimal not to punish the agent for bad (i.e., no) performance, but the investors should reward him for bad performance due to screening considerations. We leave this for future research.

6 Concluding Remarks

We study a dynamic agency problem where the agent can privately save. When ruling out private savings, previous studies (Rogerson (1985), Sannikov (2006), etc.) derive a front-loaded, performance-sensitive compensation flow in the optimal contract. In contrast, the optimal wage process in this paper becomes back-loaded, and relatively insensitive to performance.

Our optimal contract features a downward-rigid wage structure, and a seemingly "generous" severance pay in termination when the agent's performance is poor. Both patterns, which are commonly observed in today's compensation packages, have received wide criticism due to their "suboptimality" in providing incentives efficiently. Therefore, this paper delivers a general message that, under realistic contracting frictions—such as private savings and non-contractible myopic action loss studied in this paper—certain seemingly inefficient contracting features can indeed be optimal.

We solve the optimal contracting with private savings by utilizing the binding incentive-compatibility constraint in the presence of myopic actions. There, the linearity of effort cost structure is the key. However, in justifying the non-contractibility via information acquisition costs in Section 5.3, we employ a proof method which allows the agent's cost structure to be convex, and show the convergence result when the convexity diminishes. Therefore, our contracting result is generic in this regard.

Even though the resulting contract form—especially the strict downward-rigid wage—is specific to our particular setting, a less responsive wage pattern and a positive severance pay, which are designed to reduce the agent's deviation values, should be quite robust when the agent can privately save. Of course, the exact degree of robustness needs future theoretical work to explore more general settings, which might give further guidelines in solving the optimal contracting problems with private savings.

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A Appendix

A.1 Proof for Proposition 1

Take the zero saving policy as given. Under the preassumption that $a_t^* = p$ for all t, the agent's value process is $V_t = \mathbb{E}_t \left[\int_0^\tau e^{-rt} u(c_t)\, dt + e^{-r\tau} \frac{u(c_\tau)}{r} \right]$, and the martingale representation theorem implies that there exists a process $\{\beta_s\}$ such that

$$V_t = V_0 + \int_0^t e^{-rs} \beta_s \left(-pds + dN_s\right).$$

Now consider any feasible effort process $a = \{a_t \in \{0, p, \overline{p}\} : t \in [0, \tau)\}$. The agent's associated value process $V_t(a)$ could be written as

$$V_t(a) = V_0 + \int_0^t e^{-rs} \beta_s \left(-pds + dN_s(a_s)\right) + \int_0^t e^{-rt} \frac{b}{p} (p - a_s)\, ds$$

where $dN_s(a_s)$ has an intensity of a_s. Then

$$\begin{aligned} dV_t(a) &= e^{-rt} \beta_t \left(-pdt + dN_t(a_t)\right) - e^{-rt} \frac{b}{p} (a_t - p)\, dt \\ &= e^{-rt} (a_t - p) \left(\beta_t - \frac{b}{p} \right) dt + e^{-rt} \beta_t \left(dN_t(a_t) - a_t dt \right). \end{aligned}$$

Therefore, to implement working it must be the case that $(a_t - p)\left(\beta_t - \frac{b}{p}\right) \leq 0$ for both $a_t = 0$ and $a_t = \overline{p}$. This implies that $\beta_t = \frac{b}{p}$, a binding incentive-compatibility constraint. Q.E.D.

A.2 Appendix for Section 4.1

A.2.1 Production Stage

We use i to indicate the number of cashflows remaining. To start, we construct $J^0(W, M)$. When $i = 0$, there is no future cashflows, and the firm is obsolete. Based on the definition of $J^L(W)$ in (10), we have

$$J^0(W, M) = \begin{cases} J^L(W) & \text{if } W \geq \frac{U(M)}{r} \\ -\infty & \text{otherwise} \end{cases}.$$

It is clear that $J^0(W, M)$ satisfies all conditions in Proposition 3. Now consider $i \geq 1$. The next lemma translates Proposition 3 to the corresponding properties of j^{i-1}.[26]

Lemma 1 *For the wage-setting stage value function j^{i-1}, we have the following properties:*

1. $j_w^{i-1} \geq -\frac{1}{\underline{\gamma}}$, and $\frac{1}{\gamma r m} < j_m^{i-1} \leq \frac{1}{\gamma r \underline{\gamma}}$.
2. $j_{ww}^{i-1} < 0$, $j_{mm}^{i-1} < 0$, $j_{wm}^{i-1} > 0$ and $j_{ww}^{i-1} j_{mm}^{i-1} - \left(j_{wm}^{i-1}\right)^2 \geq 0$. Therefore $j^{i-1}(w, m)$ is concave.
3. $\frac{1}{\gamma r} j_{ww}^{i-1} + j_{wm}^{i-1} < 0$, $\frac{1}{\gamma r} j_w^{i-1} + j_m^{i-1} \geq 0$, and $\frac{1}{\gamma r} j_w^{i-1}\left(\frac{b}{r}, m\right) + j_m^{i-1}\left(\frac{b}{r}, m\right) = 0$.

[26] Strictly speaking, here all the second-order derivatives—j_{ww}, j_{wm}, and j_{mm}—are in the weak sense (in a Soboslov space) which allows for (finite) discontinuities, and the integration-by-parts formula still holds. To be precise, in the production stage $\widetilde{j}^i$ is a mollified version of j^{i-1}, which makes everything smooth; but the wage-setting stage only keeps the first-order smoothness (hence for the 2^{nd}-order derivatives there will be a discontinuity on $M^*(W)$). However, because the first-order derivatives are continuous, the negative definiteness of Hessian matrix is sufficient for the concavity.

Our analysis crucially depends on the following linear transformation,

$$\begin{cases} w = W - \frac{U(M)}{r} \in \left[0, \frac{b}{r}\right], \\ m = M \in \left[\underline{\gamma}, \gamma\right], \end{cases}$$

where the domain is a rectangle. Notice it is a result due to CARA specification. Let $\widetilde{j}^i(w,m) = \widetilde{J}^i(W,M)$, and $j^i(w,m) = J^i(W,M)$. Clearly $\widetilde{j}$ (j) is concave if and only if $\widetilde{J}$ (J) is concave. Note that

$$\widetilde{J}^i_W = \widetilde{j}^i_w,\ \widetilde{J}^i_M = \frac{1}{\gamma r}\widetilde{j}^i_w + \widetilde{j}^i_m, \text{ and } \widetilde{J}^i_{WM} = \frac{1}{\gamma r}\widetilde{j}^i_{ww} + \widetilde{j}^i_{wm},$$

and similar relations hold between j and J.

Without jump, $\widetilde{j}^i$ satisfies the following ODE

$$(r+p)\widetilde{j}^i(w,m) = -c(m) + p\left(Y + j^{i-1}\left(w + \frac{b}{p}, m\right)\right) + j_w(w,m)(rw - b), \tag{25}$$

and its closed-form solution is

$$\widetilde{j}^i(w,m) = \frac{r}{r+p}J^L\left(\frac{U(m)}{r}\right) + \frac{p}{r+p}[b - rw]^{1+\frac{p}{r}}\left[\int_0^w \frac{(r+p)\left(Y + j^{i-1}\left(x + \frac{b}{p}, m\right)\right)}{[b - rx]^{2+\frac{p}{r}}}dx + \frac{J^L\left(\frac{U(m)}{r}\right)}{b^{1+\frac{p}{r}}}\right] \tag{26}$$

where we use $c(m) = -rJ^L\left(\frac{U(m)}{r}\right)$. The solution in (14) in the main text is identical to (26).

Based on lemma 1, we have the following lemma for $\widetilde{j}^i$, and the results regarding $\widetilde{J}$ in Proposition 2 follows directly from this lemma.

Lemma 2 *For the production-stage value function $\widetilde{j}^i$, we have the following properties:*

1. $\widetilde{j}^i_w \geq -\frac{1}{\underline{\gamma}}$, *and* $\frac{1}{\gamma r m} < \widetilde{j}^i_m \leq \frac{1}{\gamma r \underline{\gamma}}$;
2. $\widetilde{j}^i_{ww} < 0$, $\widetilde{j}^i_{mm} < 0$, $\widetilde{j}^i_{wm} > 0$, *and* $\widetilde{j}^i_{ww}\widetilde{j}^i_{mm} - \left(\widetilde{j}^i_{wm}\right)^2 > 0$;
3. $\frac{1}{\gamma r}\widetilde{j}^i_{ww} + \widetilde{j}^i_{wm} < 0$, *and* $\frac{1}{\gamma r}\widetilde{j}^i_w\left(\frac{b}{r}, m\right) + \widetilde{j}^i_m\left(\frac{b}{r}, m\right) < 0$.

Proof. From (26), it is easy to calculate (note that $\frac{dJ^L\left(\frac{U(m)}{r}\right)}{dm} = \frac{1}{\gamma r m}$)

$$\widetilde{j}^i_m = \frac{r}{r+p}\frac{1}{\gamma r m} + \frac{p}{r+p}[b - rw]^{1+\frac{p}{r}}\left[\int_0^w (r+p)\, j^{i-1}_m\left(x + \frac{b}{p}, m\right)[b - rx]^{-2-\frac{p}{r}}dx + \frac{1}{\gamma r m}b^{-1-\frac{p}{r}}\right].$$

Notice that

$$\frac{r}{r+p} + \frac{p}{r+p}[b - rw]^{1+\frac{p}{r}}\left[\int_0^w (r+p)[b - rx]^{-2-\frac{p}{r}}dx + b^{-1-\frac{p}{r}}\right] = 1$$

which constitutes a probability measure. Since $j^{i-1}_m \in \left[\frac{1}{\gamma r m}, \frac{1}{\gamma r \underline{\gamma}}\right]$, we have $\widetilde{j}^i_m \in \left[\frac{1}{\gamma r m}, \frac{1}{\gamma r \underline{\gamma}}\right]$.

Based on (25) and (26), a direct calculation (where we use integration-by-parts formula) yields

$$\begin{aligned} & \widetilde{j}^i_w - \frac{p}{b - rw}\left(Y + j^{i-1}\left(w + \frac{b}{p}, m\right)\right) \\ = \;& -p[b - rw]^{\frac{p}{r}}\left[\int_0^w (r+p)\left(Y + j^{i-1}\left(x + \frac{b}{p}, m\right)\right)(b - rx)^{-2-\frac{p}{r}}dx + J^L\left(\frac{U(m)}{r}\right)b^{-1-\frac{p}{r}}\right] \\ = \;& p[b - rw]^{\frac{p}{r}}\left\{\left[Y + j^{i-1}\left(x + \frac{b}{p}, m\right)\right](b - rx)^{-1-\frac{p}{r}}\Big|_w^0 + \int_0^w j^{i-1}_w\left(x + \frac{p}{b}\right)[b - rx]^{-1-\frac{p}{r}}dx - J^L\left(\frac{U(m)}{r}\right)b^{-1-\frac{p}{r}}\right\}; \end{aligned}$$

therefore

$$\widetilde{j}^{i}_{w} = p\left[b-rw\right]^{\frac{p}{r}}\left\{\int_0^w j^{i-1}_w\left[b-rx\right]^{-1-\frac{p}{r}}dx+\left[Y+j^{i-1}\left(\frac{b}{p},m\right)-J^L\left(\frac{U(m)}{r}\right)\right]b^{-1-\frac{p}{r}}\right\} \quad (27)$$

$$> \left[b-rw\right]^{\frac{p}{r}}\left\{\int_0^w pj^{i-1}_w\left[b-rx\right]^{-1-\frac{p}{r}}dx+j^{i-1}_w\left(\frac{b}{p},m\right)b^{-\frac{p}{r}}+pYb^{-1-\frac{p}{r}}\right\}. \quad (28)$$

The second inequality follows from the following fact: note that $J^L\left(\frac{U(m)}{r}\right)=j^{i-1}(0,m)$, and j^{i-1} is concave, which implies that $j^{i-1}\left(\frac{b}{p},m\right)-J^L\left(\frac{U(m)}{r}\right)>j^{i-1}_w\left(\frac{b}{p},m\right)\cdot\frac{b}{p}$. Since

$$\left[b-rw\right]^{\frac{p}{r}}\left[\int_0^w p\left[b-rx\right]^{-1-\frac{p}{r}}dx+b^{-\frac{p}{r}}\right]=1 \quad (29)$$

which constitutes a probability measure, from (28) we know that $\widetilde{j}^{i}_{w}>j^{i-1}_w\geq-\frac{1}{\underline{\gamma}}$.Also, in the limiting case $w=\frac{b}{r}$, we have

$$\widetilde{j}^{i}_{w}\left(\frac{b}{r},m\right)=j^{i-1}_w\left(\frac{b}{r}+\frac{b}{p},m\right), \quad (30)$$

simply because when $w\to\frac{b}{r}$, the entire probability weights in (29) are put on $w=\frac{b}{r}$.

Now we study the second-order derivatives. It is straightforward that

$$-\widetilde{j}^{i}_{mm} = \frac{r}{r+p}\frac{1}{\gamma rm^2}+\frac{p}{r+p}\left[b-rw\right]^{1+\frac{p}{r}}\left[\int_0^w\frac{(r+p)\left[-j^{i-1}_{mm}\left(x+\frac{b}{p},m\right)\right]}{\left[b-rx\right]^{2+\frac{p}{r}}}dx+\frac{1}{\gamma rm^2}b^{-1-\frac{p}{r}}\right]$$

$$= \left[b-rw\right]^{1+\frac{p}{r}}\left[\int_0^w\frac{p\left[-j^{i-1}_{mm}\left(x+\frac{b}{p},m\right)\right]}{\left[b-rx\right]^{2+\frac{p}{r}}}dx+\left[\frac{p}{r+p}+\frac{r}{r+p}\left(\frac{b}{b-rw}\right)^{1+\frac{p}{r}}\right]\frac{b^{-1-\frac{p}{r}}}{\gamma rm^2}\right]>0$$

This shows that $\widetilde{j}^{i}$ is concave in m. For $\widetilde{j}^{i}_{ww}$, we use (25) and (28), and find that

$$-\widetilde{j}^{i}_{ww} = \frac{p}{b-rw}\left(\widetilde{j}^{i}_{w}-j^{i-1}_w\left(w+\frac{b}{p},m\right)\right)>\left[b-rw\right]^{\frac{p}{r}-1}b^{-1-\frac{p}{r}}p^2Y+$$

$$\frac{p}{b-rw}\left[\left[b-rw\right]^{\frac{p}{r}}\left[\int_0^w pj^{i-1}_w\left[b-rx\right]^{-1-\frac{p}{r}}dx+j^{i-1}_w\left(\frac{b}{p},m\right)b^{-\frac{p}{r}}\right]-j^{i-1}_w\left(w+\frac{b}{p},m\right)\right].$$

Invoking the integration-by-parts technique again, we have

$$\left[b-rw\right]^{\frac{p}{r}}\left[\int_0^w pj^{i-1}_w\left[b-rx\right]^{-1-\frac{p}{r}}dx+j^{i-1}_w\left(\frac{b}{p},m\right)b^{-\frac{p}{r}}\right]$$

$$= j^{i-1}_w\left(w+\frac{b}{p},m\right)+\left[b-rw\right]^{\frac{p}{r}}\int_0^w\left(-j^{i-1}_{ww}\right)\left[b-rx\right]^{-\frac{p}{r}}dx,$$

and therefore

$$-\widetilde{j}^{i}_{ww}>\left[b-rw\right]^{\frac{p}{r}-1}b^{-1-\frac{p}{r}}p^2Y+p\left[b-rw\right]^{\frac{p}{r}-1}\int_0^w\left(-j^{i-1}_{ww}\right)\left[b-rx\right]^{-\frac{p}{r}}dx>0. \quad (31)$$

Shortly we will need a stronger estimate for the global concavity of $\widetilde{j}$. According to (15), $Y>\frac{1}{\gamma r}\left[\frac{\gamma}{\underline{\gamma}}-1\right]^2$, and

$$-\widetilde{j}^{i}_{ww}>\left[b-rw\right]^{\frac{p}{r}-1}\left[\int_0^w p\left(-j^{i-1}_{ww}\right)\left[b-rx\right]^{-\frac{p}{r}}dx+\frac{p^2}{b^2\gamma r}\left[\frac{\gamma}{\underline{\gamma}}-1\right]^2b^{-\frac{p}{r}+1}\right]$$

Finally, we calculate

$$\widetilde{j}^i_{wm} = \frac{\partial}{\partial m}\widetilde{j}^i_w = [b-rw]^{\frac{p}{r}}\left[\int_0^w p j^{i-1}_{wm}[b-rx]^{-1-\frac{p}{r}}dx + \frac{p}{b}\left[j^{i-1}_m\left(\frac{b}{p},m\right) - \frac{1}{\gamma rm}\right]b^{-\frac{p}{r}}\right]; \qquad (32)$$

it immediately implies that $\widetilde{j}^i_{wm} \geq 0$, because $j^{i-1}_{wm} \geq 0$ and $j^{i-1}_m\left(\frac{b}{p},m\right) > j^{i-1}_m(0,m) = \frac{1}{\gamma rm}$.

Now we show that $\widetilde{j}^i$, in fact, is globally concave, which requires that $\widetilde{j}^i_{ww}\widetilde{j}^i_{mm} > \left(\widetilde{j}^i_{wm}\right)^2$. To show this, we invoke the Cauchy-Schwartz inequality. Observe that the terms other than the integral in $\widetilde{j}^i_{ww}$, $\widetilde{j}^i_{mm}$, and $\widetilde{j}^i_{wm}$ are $\frac{p^2}{b^2\gamma r}\left[\frac{\gamma}{\underline{\gamma}}-1\right]^2$, $\left[\frac{p}{r+p}+\frac{r}{r+p}\left(\frac{b}{b-rw}\right)^{1+\frac{p}{r}}\right]\frac{1}{\gamma rm^2} \geq \frac{1}{\gamma rm^2}$, and $\frac{p}{b}\left[j^{i-1}_m\left(\frac{b}{p},m\right)-\frac{1}{\gamma rm}\right]$ respectively; and we have

$$\begin{aligned}
&\frac{p^2}{b^2\gamma r}\left[\frac{\gamma}{\underline{\gamma}}-1\right]^2\left[\frac{p}{r+p}+\frac{r}{r+p}\left(\frac{b}{b-rw}\right)^{1+\frac{p}{r}}\right]\frac{1}{\gamma rm^2} \\
> \;& \frac{p^2}{b^2\gamma^2 r^2}\left[\frac{1}{\underline{\gamma}}-\frac{1}{m}\right]^2 > \frac{p^2}{b^2}\left[j^{i-1}_m\left(\frac{b}{p},m\right)-\frac{1}{\gamma rm}\right]^2.
\end{aligned}$$

Then, the standard Cauchy-Schwartz argument yields that

$$\begin{aligned}
\widetilde{j}^i_{ww}\widetilde{j}^i_{mm} &> [b-rw]^{\frac{2p}{r}}\left[\int_0^w p\left(j^{i-1}_{ww}j^{i-1}_{mm}\right)^{\frac{1}{2}}[b-rx]^{-1-\frac{p}{r}}dx + \frac{p}{b}\left[j^{i-1}_m\left(\frac{b}{p},m\right)-\frac{1}{\gamma rm}\right]b^{-\frac{p}{r}}\right]^2 \\
&> [b-rw]^{\frac{2p}{r}}\left[\int_0^w p\left|j^{i-1}_{wm}\right|[b-rx]^{-1-\frac{p}{r}}dx + \frac{p}{b}\left[j^{i-1}_m\left(\frac{b}{p},m\right)-\frac{1}{\gamma rm}\right]b^{-\frac{p}{r}}\right]^2 \geq \left(\widetilde{j}^i_{wm}\right)^2
\end{aligned}$$

where we use the fact that j^{i-1} is concave. By taking $i \to \infty$, this argument implies that a "tighter" sufficient condition for the concavity of J is $\frac{p^2 Y}{b^2}\frac{1}{\gamma rm^2} \geq \frac{p^2}{b^2}\left[j_m\left(\frac{b}{p},m\right)-\frac{1}{\gamma rm}\right]^2$, or

$$Y > \max_{m\in[\underline{\gamma},\gamma]}\gamma r\left[mj_m\left(\frac{b}{p},m\right)-\frac{1}{\gamma r}\right]^2.$$

Finally we show the property 3. According to (15), $Y > \frac{b}{p\underline{\gamma}}$. Utilizing (32) and (31), and since $j^{i-1}_m\left(\frac{b}{p},m\right) - \frac{1}{\gamma rm} < \frac{1}{\gamma r}\frac{1}{\underline{\gamma}}$, we have

$$\begin{aligned}
\frac{1}{\gamma r}\widetilde{j}^i_{ww}+\widetilde{j}^i_{wm} &< [b-rw]^{\frac{p}{r}-1}\left[\int_0^w p\left(\frac{1}{\gamma r}j^{i-1}_{ww}\right)[b-rx]^{-\frac{p}{r}}dx\right] + [b-rw]^{\frac{p}{r}}\left[\int_0^w pj^{i-1}_{wm}[b-rx]^{-1-\frac{p}{r}}dx\right] \\
&= [b-rw]^{\frac{p}{r}}\left[\int_0^w p\left(\frac{1}{\gamma r}j^{i-1}_{ww}+j^{i-1}_{wm}\right)[b-rx]^{-1-\frac{p}{r}}dx\right] + \\
&\quad [b-rw]^{\frac{p}{r}-1}\left[\int_0^w p\left(\frac{1}{\gamma r}j^{i-1}_{ww}\right)[b-rx]^{-\frac{p}{r}}dx\right] - [b-rw]^{\frac{p}{r}}\left[\int_0^w p\left(\frac{1}{\gamma r}j^{i-1}_{ww}\right)[b-rx]^{-1-\frac{p}{r}}dx\right]
\end{aligned}$$

The first item is negative since $\frac{1}{\gamma r}j^{i-1}_{ww}+j^{i-1}_{wm} < 0$. Because $j^{i-1}_{ww} < 0$, and for $x < w$ we have

$$[b-rw]^{\frac{p}{r}-1}[b-rx]^{-\frac{p}{r}} > [b-rw]^{\frac{p}{r}}[b-rx]^{-1-\frac{p}{r}},$$

the second item is negative too. Therefore $\frac{1}{\gamma r}\widetilde{j}^i_{ww}+\widetilde{j}^i_{wm} < 0$.

The second inequality in property 3 says that $\widetilde{J}^i_M\left(\frac{U(M)+b}{r}+\frac{b}{p},M\right) < 0$. To show this, When $W = \frac{U(M)+b}{r}$, take derivative w.r.t M on equation (13), one finds that

$$r\widetilde{J}^i_M = -c'(M) + p\left(J^{i-1}_M\left(\frac{U(M)+b}{r}+\frac{b}{p},M\right)-\widetilde{J}^i_M\right)+\frac{1}{\gamma}\widetilde{J}^i_W \Rightarrow \widetilde{J}^i_M = \frac{1}{\gamma(r+p)}\left(\frac{1}{M}+\widetilde{J}^i_W\right)$$

where we use $J_M^{i-1}\left(\frac{U(M)+b}{r}+\frac{b}{p},M\right)=0$ (Proposition 3, Property 3), and $-c'(M)=\frac{1}{\gamma M}$. However, we have shown that $\widetilde{J}_W^i\left(\frac{U(M)}{r}+\frac{b}{r},M\right)=j_w^{i-1}\left(\frac{b}{r}+\frac{b}{p},m\right)$ in (30). Now, as verified in the next wage-setting stage, investors raise the agent's wage, and as a result there exists $m^*<m$ so that

$$\begin{aligned} j_w^{i-1}\left(\frac{b}{r}+\frac{b}{p},m\right) &= -\gamma r j_m^{i-1}\left(\frac{b}{r}+\frac{U(m)-U(m^*)}{r}+\frac{b}{p},m^*\right) \\ &< -\gamma r\frac{1}{\gamma r m^*}\le -\frac{1}{m}. \end{aligned}$$

Therefore $\widetilde{J}_W^i<-\frac{1}{M}$, and $\widetilde{J}_M^i\left(\frac{U(M)+b}{r},M\right)<0$. Q.E.D. ■

A.2.2 Wage-setting Stage

First we show the zero marginal cost brought on by future termination at $W=\frac{b+U(M)}{r}$. Notice that rasing wage at $W=\frac{b+U(M)}{r}$ is equivalent to setting w below $\frac{b}{r}$. Consider the policy of setting $W^*(M)$ so that $w=\frac{b}{r}-\varepsilon$. Then starting from $(W^*(M),M)$, it is easy to check that the expected discounted termination probability is $\left(\frac{r\varepsilon}{b}\right)^{\frac{r+p}{r}}$ on the path without any jumps. Under the Poisson setup, the total expected discounted termination probability—by integrating over all jumps but with the same M—is still in the order of $\varepsilon^{\frac{r+p}{r}}$; and notice that it is an upper-bound estimator, as if a jump leads to a lower M^* then the impact on the probability of future terminations is zero. Therefore reducing M has a zero marginal impact on $\varepsilon=0$ when $p>0$.

Next we present a formal construction of J^i from $\widetilde{J}^i$. Given $M^*(W)$ defined in the main text (note that $M^*(\cdot)$ might be i-dependent), we propose a transformation

$$\mathbb{T}(W,M)=(W,\min(M,M^*(W))), \tag{33}$$

and define $J^i(W,M)=\widetilde{J}^i(\mathbb{T}(W,M))$. This transformation preserves the concavity. To see this, consider any two points (W_1,M_1) and (W_2,M_2) and

$$W(\lambda)=\lambda W_1+(1-\lambda)W_2 \text{ and } M(\lambda)=\lambda M_1+(1-\lambda)M_2.$$

For $\mathbf{S}=\mathbb{T}(W(\lambda),M(\lambda))$ and $\mathbf{S}'=\lambda\mathbb{T}(W_1,M_1)+(1-\lambda)\mathbb{T}(W_2,M_2)$, both have the same W, but $\mathbf{S}$ has a larger M. Because both $\mathbf{S}$ and $\mathbf{S}'$ are in the region where $\widetilde{J}_M^i\ge 0$, we have $\widetilde{J}^i(\mathbf{S})\ge\widetilde{J}^i(\mathbf{S}')$. Therefore

$$\begin{aligned} J^i(W(\lambda),M(\lambda)) &= \widetilde{J}^i(\mathbb{T}(W(\lambda),M(\lambda)))\ge\widetilde{J}^i(\lambda\mathbb{T}(W_1,M_1)+(1-\lambda)\mathbb{T}(W_2,M_2)) \\ &\ge \lambda\widetilde{J}^i(\mathbb{T}(W_1,M_1))+(1-\lambda)\widetilde{J}^i(\mathbb{T}(W_2,M_2)) \\ &= \lambda J^i(W_1,M_1)+(1-\lambda)J^i(W_2,M_2). \end{aligned}$$

It is easy to check that the resulting $J^i(W,M)$ ($j^i(w,m)$) satisfies all properties stated in Proposition 3 (Proposition 1). For completeness, we provide several properties of j^i on the domain above the curve $M^*(W)$. Notice that

$$j^i(w,m)=J^i(W,M)=\widetilde{J}^i(W,M^*(W))=\widetilde{j}^i\left(W-\frac{U(m^*)}{r},m^*\right),$$

where $m^*=M^*>M$. By construction, $J_M^i(W,M)=\frac{1}{\gamma r}j_w^i(w,m)+j_m^i(w,m)=0$. Then utilizing the fact that $\widetilde{J}_M^i(W,M^*(W))=0$ (therefore the indirect impact on m^* (or M^*) is zero), one can easily verify that

$$\begin{aligned} j_w^i(w,m) &= \widetilde{j}_w^i\left(W-\frac{U(m^*)}{r},m^*\right) \\ j_m^i(w,m) &= -\frac{1}{\gamma r}j_w^i(w,m)=\widetilde{j}_m^i\left(W-\frac{U(m^*)}{r},m^*\right)\ge\frac{1}{\gamma r m^*}\in\left(\frac{1}{\gamma r m},\frac{1}{\gamma r\underline{\gamma}}\right] \\ \frac{1}{\gamma r}j_{ww}^i+j_{wm}^i &= J_{WM}^i(W,M)=0, \text{ and } j_{mm}^i=\frac{1}{\gamma^2 r^2}j_{ww}^i. \end{aligned}$$

A.2.3 Convergence and the Upper-First-Best States

Let $\mathcal{C}(X)$ as the set of continuous, bounded and concave functions on the convex compact set

$$X = \left\{(W, M) : M \in \left[\underline{\gamma}, \gamma\right], W \in \left[\frac{U(M)}{r}, \frac{U(M)+b}{r}\right]\right\} \subset \mathbb{R}^2.$$

We have defined an operator $\mathbb{O} : \mathcal{C}(X) \to \mathcal{C}(X)$ to construct $J^i = \mathbb{O}\left(J^{i-1}\right)$ successively. Specifically, for $J^{i-1} \in \mathcal{C}(X)$, define J^i in two steps. First,

$$\begin{aligned}
\widetilde{J}^i(W, M) &= [b - rW + U(M)]^{1+\frac{p}{r}} \left[\int_{\frac{U(M)}{r}}^{W} \frac{pY - c(M) + pJ^{i-1}\left(x + \frac{b}{p}, M\right)}{[b - rx + U(M)]^{-2-\frac{p}{r}}} dx + J^L\left(\frac{U(M)}{r}\right) b^{-1-\frac{p}{r}}\right] \\
&= \frac{rJ^L\left(\frac{U(m)}{r}\right)}{r+p} + \frac{p[b - rW + U(M)]^{1+\frac{p}{r}}}{r+p} \left[\int_{\frac{U(M)}{r}}^{W} \frac{(r+p)\left(Y + J^{i-1}\left(x + \frac{b}{p}, M\right)\right)}{[b - rx + U(M)]^{-2-\frac{p}{r}}} dx + \frac{J^{i-1}\left(\frac{U(M)}{r}, M\right)}{b^{1+\frac{p}{r}}}\right],
\end{aligned}$$

since $J^{i-1}\left(\frac{U(M)}{r}, M\right) = J^L\left(\frac{U(M)}{r}\right)$. Second, the transformation $\mathbb{T}(W, M)$ defined in (33) gives

$$J^i(W, M) = \widetilde{J}^i(\mathbb{T}(W, M)).$$

Now we show that mapping $\mathbb{O}$ satisfies Blackwell's sufficient conditions for a contraction mapping (Stokey and Lucas (1989)), which implies that there exists a unique J such that J^i converges to $J \in \mathcal{C}(X)$ uniformly.

We need to verify the monotonicity condition,

$$\mathbb{O}(f) \leq \mathbb{O}(g) \text{ if } f \leq g,\ f, g \in \mathcal{C}(X),$$

and the discounting condition

$$\mathbb{O}(f + x) \leq \mathbb{O}f + \frac{p}{r+p}x \text{ where } f \in \mathcal{C}(X), x \in \mathbb{R}$$

To see the monotonicity condition, decompose $\mathbb{O}$ into $\mathbb{O}_1$ (from J^{i-1} to $\widetilde{J}^i$) and $\mathbb{O}_2$ (from $\widetilde{J}^i$ to J^i). If $f \leq g$, then $\mathbb{O}_1 f \leq \mathbb{O}_1 g$. Fix W, and let M_f^* and M_g^* be the corresponding wage-setting curves. Clearly if $M < \min\left(M_f^*, M_g^*\right)$ then $\mathbb{O}_2 f \leq \mathbb{O}_2 g$ holds. If $M > \max\left(M_f^*, M_g^*\right)$,

$$\mathbb{O}_2(f)(W, M) = \mathbb{O}_1(f)\left(W, M_f^*\right) \leq \mathbb{O}_1(g)\left(W, M_f^*\right) \leq \mathbb{O}_2(g)\left(W, M_g^*\right) = \mathbb{O}_2(g)(W, M)$$

Finally consider M sits between M_f^* and M_g^*. W.l.o.g, consider $M_f^* < M_g^*$. Then

$$\mathbb{O}_2(f)(W, M) = \mathbb{O}_1(f)\left(W, M_f^*\right) \leq \mathbb{O}_1(g)\left(W, M_f^*\right) \leq \mathbb{O}_1(g)(W, M) = \mathbb{O}_2(g)(W, M)$$

where the third inequality uses the fact that $\mathbb{O}_1(g)$ is concave, $M_f^* < M < M_g^*$, and M_g^* attains the maximum. The second discounting condition is straightforward.

Note that we have focused on the case $M > \underline{\gamma}$; however, the previous construction also applies to the line with $M = \underline{\gamma}$ and $W < \frac{U(\underline{\gamma})+b}{r}$. To complete the construction of J, we derive the value function for the upper-first-best states where $M = \underline{\gamma}$ and $W \geq \frac{U(\underline{\gamma})+b}{r}$. Since the agent is risk-neutral, one particular solution has the agent consume $\frac{1}{\underline{\gamma}}\left(W - \frac{U(\underline{\gamma})+b}{r}\right)$ whenever $W \geq \frac{U(\underline{\gamma})+b}{r}$; afterwards and the state-pair stays at $\left(\frac{U(\underline{\gamma})+b}{r}, \underline{\gamma}\right)$ without jumps, and the agent obtains $\frac{b}{p\underline{\gamma}} < Y$ whenever a cashflow occurs. Based on (13) it is easy to show that in this region

$$J\left(W, \underline{\gamma}\right) = J\left(\frac{U\left(\underline{\gamma}\right)+b}{r}, \underline{\gamma}\right) - \frac{1}{\underline{\gamma}}\left(W - \frac{U\left(\underline{\gamma}\right)+b}{r}\right) = \frac{pY}{r} - \frac{u^{-1}(rW)}{r},$$

which is the first-best result when K, the maximum number of cashflows generated by the agent, is ∞. When K is finite, we can just replace $\frac{pY}{r}$ with $\frac{pY}{r}\left[1 - \left(\frac{p}{r+p}\right)^K\right]$ in the above equation.

A.3 Appendix for Section 4.2

Following Sannikov (2006), we denote the investors' concave value function as $f(W)$, and continuation payoff W follows

$$dW = (rW - u(c^*))\,dt + \frac{b}{p}(dN_t - pdt),$$

where c^* solves the investors' HJB equation

$$rf(W) = \max_{c\geq 0}\left\{pY - c + p\left[f\left(W + \frac{b}{p}\right) - f(W)\right] + f'(W)[rW - u(c) - b]\right\}. \quad (34)$$

Clearly, due to the risk-neutrality for a sufficiently high consumption level, similar to the previous discussion there is an absorbing first-best state for $W \geq \frac{U(\underline{\gamma})+b}{r}$, and $f'(W) = \frac{1}{\underline{\gamma}}$. Note that in Sannikov (2006) the upper-absorbing state corresponds to the case where the wealth effect becomes extreme, and the firm is terminated. The difference is purely due to different utility specifications.

In the lower region where $f'(W) > -\frac{1}{\underline{\gamma}}$, it is easy to show that the optimal wage policy, as a function of W, is

$$c^* = \begin{cases} \frac{1}{\gamma}\ln\left(\frac{-1}{f'(W)\gamma}\right) & \text{when } f'(W) < -\frac{1}{\gamma} \\ 0 & \text{otherwise} \end{cases}.$$

This policy can be understood as follows. In (34), paying one more dollar of wage has a unit marginal cost; and on the benefit side, it reduces the agent's continuation payoff by $u'(c)$, so the marginal benefit is $-f'(W)u'(c)$. The above policy equates the marginal cost with the marginal benefit whenever possible. As f is concave, c^* will bind at zero for low W's, which reflects the fact that when the inefficient termination (once $W = 0$) is close, the marginal benefit of reducing continuation payoff $-f'(W)u'(c)$ either is small, or even becomes negative.

A.4 Appendix for Section 4.3

A.4.1 Appendix for 4.3.2

Recall that by implementing the myopic effort, investors suffer a non-contractible loss Δ. Therefore we have (recall that $\overline{p} = p + \epsilon$),

$$\begin{aligned}
\mathbb{E}_t e^{rt} dG_t &= \left[-rJ - c(M) + \overline{p}\left(Y + \left[J\left(W + \beta_t, M^1_{t+}\right) - J\right]\right) + J_W\left(rW - U(M) + \frac{b\epsilon}{p} - \beta_t \overline{p}\right)\right] dt + J_M\left(M^0_{t+} - M\right) - \Delta dt \\
&\leq \left[-rJ - c(M) + \overline{p}\left(Y + \left[J\left(W + \beta_t, M\right) - J\right]\right) + J_W\left(rW - U(M) + \frac{b\epsilon}{p} - \beta_t \overline{p}\right)\right] dt + J_M \overline{p}\left(M - \underline{\gamma}\right) dt - \Delta dt \\
&\leq \left[-rJ - c(M_t) + \overline{p}\left(Y + \left[J\left(W + \frac{b}{p}, M\right) - J\right]\right) + J_W\left(rW - U(M) - b\right)\right] dt + J_M \overline{p}\left(M - \underline{\gamma}\right) dt - \Delta dt \\
&= \epsilon\left[Y + \left[J\left(W + \frac{b}{p}, M\right) - J\right] + J_M\left(M - \underline{\gamma}\right)\right] dt + J_M p\left(M - \underline{\gamma}\right) dt - \Delta dt. \quad (35)
\end{aligned}$$

The second inequality is derived as follows. From (20), we know that $M^0_{t+} \leq \frac{M - M^1_{t+}\overline{p}dt}{1 - \overline{p}dt}$, therefore

$$\overline{p}J\left(W + \beta_t, M^1_{t+}\right) + J_M\left(M^0_{t+} - M\right) \leq \overline{p}J\left(W + \beta_t, M^1_{t+}\right) dt + J_M \cdot \left(\frac{M - M^1_{t+}\overline{p}dt}{1 - \overline{p}dt} - M\right).$$

The RHS reflects a trade-off: reducing M^1_{t+} can increase the value without jump (a higher M^0_{t+}), but potentially hurt the value after the jump if $J_M\left(W + \beta_t, M^1_{t+}\right) > 0$. The derivative w.r.t. M^1_{t+} is $\overline{p}\left[J_M\left(W + \beta_t, M^1_{t+}\right) - J_M(W, M)\right] dt$. Because $J_{WM} < 0$, the optimal M^1_{t+} takes a value $\widehat{M} \in \left[\underline{\gamma}, \min\left(M_t, M^*\left(W + \beta_t\right)\right)\right]$. Therefore

$$\begin{aligned}
\overline{p}J\left(W + \beta_t, M^1_{t+}\right) dt + J_M\left(M^0_{t+} - M\right) &\leq \overline{p}J\left(W + \beta_t, \widehat{M}\right) dt + J_M\left(\frac{M - \widehat{M}\overline{p}dt}{1 - \overline{p}dt} - M\right) \\
&\leq \overline{p}J\left(W + \beta_t, M\right) dt + J_M \overline{p}\left(M - \underline{\gamma}\right) dt,
\end{aligned}$$

where last inequality is obtained by replacing $\widehat{M}$ with M in the first term (as $\widehat{M} < M$ and $J_M \geq 0$), and replacing $\widehat{M}$ with $\underline{\gamma}$ in the second term. The third inequality in (35) is due to the fact that J is concave, so the problem

$$\max_{\beta_t \geq \frac{b}{p}} \overline{p}\left[J\left(W+\beta_t, M\right)-J\right]-J_W \beta_t \overline{p}$$

picks up the corner solution $\frac{b}{p}$.

In (35), we take ϵ to be arbitrarily small. Because $J_{WM} < 0$, when M is fixed, J_M attains the maximum when $W = \frac{U(M)}{r}$. Therefore a sufficient condition, which can be verified easily ex-post, is

$$\Delta > \max_{M \in \left[\underline{\gamma}, \gamma\right]} p\left(M-\underline{\gamma}\right) J_M\left(\frac{U\left(M\right)}{r}, M\right).$$

There is another thorny technical issue. Under implementing myopic effort, theoretically investors can deliver (W, M) below the termination curve $W = \frac{U(M)}{r}$. To see this, using the argument in footnote 14, the agent's continuation utility from consumption is $\mathbb{E}\left[\int_0^{\infty} e^{-rt} U\left(M_t\right) dt\right] \geq \frac{U(M)}{r}$; but since the agent incurs certain myopic effort cost (which is bounded by $\frac{b}{pr}\epsilon$) in equilibrium, the lower bound estimator for W is $W \geq \frac{U(M)}{r} - \frac{b}{pr}\epsilon$. However, when ϵ is sufficiently small, this possibility is immaterial. Also, the main concern is that whether the following policy is profitable: instead of immediate termination, investors asks the agent to take myopic effort in the firm, and potentially fire the agent in another state-pair $\left(\frac{U(M')}{r}, M'\right)$ where $M' > M$. Clearly, because it is only possible when $a = \overline{p}$, a sufficiently high loss Δ leads to a suboptimality relative to an immediate liquidation at $\left(\frac{U(M)}{r}, M\right)$. In fact, based on this intuition He (2007c) derives a lower bound as a function of ϵ for the sneaking loss Δ. For details and another sufficient condition that is independent of Δ, see He (2007c).

A.5 Appendix for Section 5.1

Let $g(\cdot) = u'(\cdot)$; then $U(M) = u\left(g^{-1}(M)\right)$, and $l(M) = \frac{U(M)}{r}$. It is easy to check that

$$l'(m) = \frac{m}{ru''} < 0, l''(m) = \frac{u'u''' - \left(u''\right)^2}{r\left(-u''\right)^3} > 0,$$

where the second inequality, which is equivalent to $u''' > \frac{\left(u''\right)^2}{u'} > 0$, is imposed as assumption. Here $J^L\left(\frac{U(M)}{r}\right) = -\frac{g^{-1}(M)}{r}$; so $\frac{dJ^L}{dM} = \frac{1}{r(-u'')} > 0$ and $\frac{d^2J^L}{dM^2} = \frac{1}{r}\frac{u'''}{(u'')^3} < 0$. Similar to the previous analysis, define a *nonlinear* transformation (omitting the superscript i, if any)

$$J(W, M) = j\left(W - \frac{U(M)}{r}, M\right) = j(w, m).$$

Therefore

$$J_W = j_w, J_M = \frac{m}{r\left(-u''\right)} j_w + j_m$$

and

$$J_{WW} = j_{ww}, J_{WM} = \frac{m}{r\left(-u''\right)} j_{ww} + j_{wm}, J_{MM} = -j_w l'' + j_{ww}\left(l'\right)^2 - 2l' j_{wm} + j_{mm}$$

and

$$J_{WW} J_{MM} - \left(J_{WM}\right)^2 = -j_{ww} j_w l'' + j_{ww} j_{mm} - \left(j_{wm}\right)^2.$$

The following two lemma give corresponding results for Proposition 3 and Proposition 2.

Lemma 3 *For the setting-wage stage value function $j^{i-1}(w, m)$, we have*

1. $j_w^{i-1} \geq -\frac{1}{\underline{\gamma}}$, *and* $\frac{1}{r(-u'')} < j_m^{i-1} \leq \frac{m}{r(-u'')\underline{\gamma}}$;

2. $J_{WW}^{i-1} < 0$, $J_{MM}^{i-1} \leq 0$, *and* $J_{WW}^{i-1} J_{MM}^{i-1} - \left(J_{WM}^{i-1}\right)^2 \geq 0$. *It implies that* J^{i-1} *is concave;*

3. $J_{WM}^{i-1} = \frac{m}{r(-u'')} j_{ww}^{i-1} + j_{wm}^{i-1} \leq 0$, *and* $J_M^{i-1} = \frac{m}{r(-u'')} j_w^{i-1} + j_m^{i-1} \geq 0$. *We have* $J_M^{i-1}\left(\frac{U(M)+b}{r}, M\right) = 0$.

Lemma 4 *For the production stage value function* $\widetilde{j}^i(w, m)$*, we have*

1. $\widetilde{j}_w^i \geq -\frac{1}{\underline{\gamma}}$, *and* $\frac{1}{r(-u'')} < \widetilde{j}_m^i \leq \frac{m}{r(-u'')\underline{\gamma}}$;

2. $\widetilde{J}_{WW}^i < 0$, $\widetilde{J}_{MM}^i < 0$, *and* $\widetilde{J}_{WW}^i \widetilde{J}_{MM}^i - \left(\widetilde{J}_{WM}^i\right)^2 > 0$. *It implies that* $\widetilde{J}^i$ *is concave;*

3. $\widetilde{J}_{WM}^i = \frac{m}{r(-u'')} \widetilde{j}_{ww}^i + \widetilde{j}_{wm}^i < 0$. *We have* $\widetilde{J}_M^i\left(\frac{U(M)+b}{r}, M\right) < 0$.

For detailed proofs, and other requirements when singularity is involved (e.g., the case where $u'' \to 0$ when $u' \to \underline{\gamma}$, rather than the case where there exists $\bar{c}$ such that $u''(\bar{c}) < 0$ and $u''(\bar{c}+) = 0$ as assumed in the main text), see He (2007c).

A.6 Appendix for Section 5.2

We again construct $J^{RP}(W, M)$ recursively. The following lemma lists the properties of $j^{RP,i-1}(W, M)$. In property 4, $\underline{w}^{i-1}(m)$ is the renegotiation curve discussed in the main text, and $W^{RP,i-1,*}(m)$ is the wage-setting curve similar to the definition in (16).

Lemma 5 *For the wage-setting stage value function* $j^{RP,i-1}(w, m)$*, we have the following properties:*

1. $-\frac{1}{\underline{\gamma}} \leq j_w^{RP,i-1} \leq 0$, $j_m^{RP,i-1} > \frac{1}{\gamma r m}$, and $0 \leq \frac{1}{\gamma r} j_w^{RP,i-1} + j_m^{RP,i-1} \leq \frac{1}{\gamma r m}$.

2. $j_{ww}^{RP,i-1} < 0$, $j_{mm}^{RP,i-1} < 0$, $j_{wm}^{RP,i-1} > 0$, and $j_{ww}^{RP,i-1} j_{mm}^{RP,i-1} - \left(j_{wm}^{RP,i-1}\right)^2 \geq 0$. Therefore $j^{RP,i-1}(w, m)$ is concave.

3. $\frac{1}{\gamma r} j_{ww}^{RP,i-1} + j_{wm}^{RP,i-1} \leq 0$, $\frac{1}{\gamma r} j_w^{RP,i-1}\left(\frac{b}{r}, m\right) + j_m^{RP,i-1}\left(\frac{b}{r}, m\right) \geq 0$, and $\frac{1}{\gamma r} j_w^{RP,i-1}\left(\frac{b}{r}, m\right) + j_m^{RP,i-1}\left(\frac{b}{r}, m\right) = 0$.

4. $\underline{w}^{i-1}(m) < W^{RP,i-1,*}(m) - \frac{U(m)}{r} \leq \frac{b}{r}$, $\underline{w}^{i-1\prime}(m) \geq 0$.

Consider the production stage in i^{th} subperiod. There exists a curve $\underline{w}^i(m)$ such that $\widetilde{j}^{RP,i}$ takes the value $J^L\left(\frac{U(m)}{r}\right)$, and $\widetilde{j}_w^{RP,i} = 0$ on this curve. Similar to (27), one can check that

$$\widetilde{j}_w^{RP,i}(w, m) = p\,[b - rw]^{\frac{p}{r}} \left\{ \int_{\underline{w}^i(m)}^{w} j_w^{RP,i-1}\,[b - rx]^{-1-\frac{p}{r}}\,dx + \frac{\widehat{Y} + j^{RP,i-1}\left(\underline{w}^i(m) + \frac{b}{p}, m\right) - J^L\left(\frac{U(m)}{r}\right)}{\left(b - r\underline{w}^i(m)\right)^{\frac{p}{r}}} \right\},$$

where $\widehat{Y} = \frac{pY - rL}{p}$. In the spirit of renegotiation-proof contract, at $\underline{w}^i(m)$, $\widetilde{j}_w^{RP,i} = p\left[Y + j^{RP,i-1}\left(\underline{w}^i(m) + \frac{b}{p}, m\right) - J^L\left(\frac{U(m)}{r}\right)\right]$ is zero at $\underline{w}^i(m)$. Therefore we define

$$\underline{w}^i(m) = \inf\left\{0 \leq x \leq \frac{b}{r} : \left[\widehat{Y} + j^{RP,i-1}\left(x + \frac{b}{p}, m\right) - J^L\left(\frac{U(m)}{r}\right)\right] = 0\right\}.$$

We assume that $\underline{w}^i(m) < \frac{b}{r}$, which holds when L is relatively large, or $\widehat{Y}$ is relatively small. Under this condition, we can show that $\underline{w}^i(m) < W^{RP,i-1,*}(m) - \frac{U(m)}{r}$. For instance, when $M = \underline{\gamma}$, for $\underline{W}(\underline{\gamma})$ to take a value below the wage-setting point $W^*(\underline{\gamma}) = \frac{U(\underline{\gamma})+b}{r}$, we require that the investors' value at termination is greater than their value at the upper-first-best boundary point, i.e., $J^L\left(\frac{U(\underline{\gamma})}{r}\right) > J\left(\frac{U(\underline{\gamma})+b}{r}, \underline{\gamma}\right) \Leftrightarrow rL - pY > -\frac{b}{\underline{\gamma}} \Leftrightarrow \widehat{Y} < \frac{b}{p\underline{\gamma}}$. We have the following lemma for $\widetilde{j}^{RP,i}$.

Lemma 6 *For the production-stage value function $\widetilde{j}^{RP,i}(w,m)$, we have*

1. $\widetilde{j}_w^{RP,i} < 0$, $\widetilde{j}_m^{RP,i} > \frac{1}{\gamma rm}$, *and* $\frac{1}{\gamma r}\widetilde{j}_w^{RP,i} + \widetilde{j}_m^{RP,i} \leq \frac{1}{\gamma rm}$.

2. $\widetilde{j}_{ww}^{RP,i} < 0$, $\widetilde{j}_{mm}^{RP,i} < 0$, $\widetilde{j}_{wm}^{RP,i} > 0$, *and* $\widetilde{j}_{ww}^{RP,i}\widetilde{j}_{mm}^{RP,i} - \left(\widetilde{j}_{wm}^{RP,i}\right)^2 > 0$. *Therefore* $\widetilde{j}^{RP,i}(w,m)$ *is concave.*

3. $\frac{1}{\gamma r}\widetilde{j}_{ww}^{RP,i} + \widetilde{j}_{wm}^{RP,i} \leq 0$, $\frac{1}{\gamma r}\widetilde{j}_w^{RP,i}\left(\frac{b}{r}, m\right) + \widetilde{j}_m^{RP,i}\left(\frac{b}{r}, m\right) < 0$.

4. $\underline{w}^{i\prime}(m) \geq 0$.

For detailed proofs, see He (2007c). When L is small (for instance, $L = 0$), $\underline{w}(m)$ and $W^{RP,*}(m) - \frac{U(m)}{r}$ both bind at $\frac{b}{r}$. At this point, without success the agent stays at that point, and after a jump the agent is promoted to another point with lower m (higher wage). Because termination is extremely inefficient ($pY > \frac{b}{\underline{\gamma}}$ so keeping the project alive is better off always), termination will be off-equilibrium.

A.7 Appendix for Section 5.3

Since dQ information does not add value in implementing $a = 0$ or $a = \overline{p}$, both actions are suboptimal. In implementing $a = p$, recall that

$$dW_t = (rW - U(M))\,dt + \beta_t (dN_t - pdt) + x_t (dQ_t - \mu dt),$$

where $\beta_t = \frac{b}{p} + k_t > \frac{b}{p}$, and $x_t \geq \frac{k_t \epsilon}{\Delta}$. Note that $k_t \geq 0$; otherwise the agent shirks (without affecting the dQ_t performance). Suppose that the evolution of M can be written as,

$$dM_t = M_{t+dt}^1 dN_t + \left(M_{t+dt}^0 - M_t\right) + x_t^M dZ_t;$$

any drift is absorbed in the term of $M_{t+dt}^0 - M_t$. Since J is concave, to mitigate the second-order effect $\frac{k_t^2\epsilon^2\sigma^2}{\Delta^2}J_{WW} + 2x_t^M\frac{k_t\epsilon\sigma}{\Delta}J_{WM} + \left(x_t^M\right)^2 J_{MM}$, the optimal x_t^M is $-\frac{k_t\epsilon\sigma J_{WM}}{\Delta J_{MM}}$, which gives a lower bound estimate of the second order loss due to extra loadings on dQ information:

$$\frac{k_t^2\epsilon^2\sigma^2}{\Delta^2}\left(\frac{J_{WW}J_{MM} - J_{WM}}{J_{MM}}\right) = \frac{k_t^2\epsilon^2\sigma^2}{\Delta^2}\left(\frac{j_{ww}j_{mm} - j_{wm}}{\frac{1}{\gamma^2 r^2}j_{ww} + \frac{2}{\gamma r}j_{wm} + j_{mm}}\right).$$

Now for investors, the value still can be written as $J^{YQ} = \mathbb{E}[G_\tau]$ with G_τ defined in (18), under some optimal contract that incorporates both the Y and Q information. As $G_0 = J$, $J^{YQ} - J = \mathbb{E}[G_\tau - G_0]$, the net gain by incorporating Q performance is

$$\begin{aligned} J^{YQ} - J &= \mathbb{E}\left\{\begin{array}{c} \int_0^\tau e^{-rt}\left[\left[-rJ - c(M_t) + p\left(Y + \left[J\left(W + \frac{b}{p}, M_{t+}^1\right) - J\right]\right) + J_W(rW - U(M) - b)\right]dt\right] \\ +e^{-rt}J_M\left(M_{t+}^0 - M_t\right) + e^{-rt}\frac{k_t^2\epsilon^2\sigma^2}{\Delta^2}\left(\frac{J_{WW}J_{MM} - J_{WM}}{J_{MM}}\right)dt \end{array}\right\} \\ &= \mathbb{E}\left\{\begin{array}{c} \int_0^\tau e^{-rt}\left[\left[-rJ - c(M_t) + p\left(Y + \left[J\left(W + \frac{b}{p}, M\right) - J\right]\right) + J_W(rW - U(M) - b)\right]dt\right] \\ +e^{-rt}\left\{p\left[J\left(W + \frac{b}{p}, M_{t+}^1\right) - J\left(W + \frac{b}{p}, M\right)\right]dt + J_M\left(M_{t+}^0 - M_t\right)\right\} \\ +e^{-rt}\frac{k_t^2\epsilon^2\sigma^2}{\Delta^2}\left(\frac{J_{WW}J_{MM} - J_{WM}}{J_{MM}}\right)dt \end{array}\right\} \end{aligned} \quad (36)$$

where τ could take the value ∞. Note that due to construction, the first line is zero.

For the second line, using the same argument as in Section A.4.1 and footnote 20, we let $M^0_{t+dt} - M = xdt$, and then it becomes

$$\begin{aligned} & p\left[J\left(W+\frac{b}{p}, M-\frac{x}{p}\right)-J\left(W+\frac{b}{p}, M\right)\right] dt + J_M \cdot xdt \\ \leq & \left[J_M\left(W, M\right)-J_M\left(W+\frac{b}{p}, M\right)\right]\left(M^0_{t+}-M\right) \end{aligned} \tag{37}$$

because J is concave in M. Note that $J_M(W,M) - J_M\left(W+\frac{b}{p}, M\right) > 0$; it just captures that idea that investors gains by reducing wage without jump (so $M^0_{t+} > M$). For a simpler notation, later we replace this gain in (37) by

$$J_M(W,M)\left(M^0_{t+}-M\right),$$

because its analytical property is exactly the same as the one in (37). Note that the total expected gain is bounded by the the difference between the investors' value without private savings derived in Section 4.2 and J derived in Section 4.

The third line is the cost brought on by dQ. Our main argument will be that since J is concave ($\frac{J_{WW}J_{MM}-J_{WM}}{J_{MM}}$ is negative), when $\sigma^2 \to \infty$, k_t has to be sufficiently small. Because then the shirking loss is diminishing, this limits the extent of $M^0_{t+dt} - M_t$, and therefore the contractual gain identified above goes to zero.

There are several technical issues. On the state space, when (W, M) is close enough to the upper-first-best region ($M = \underline{\gamma}$), J fails to be strict concave in the first-best region $W \geq \frac{U(\underline{\gamma})+b}{r}$. However, based on (20), we have

$$\begin{aligned} M^0_{t+} - M & \leq \overline{p}\left(M^0_{t+}-M^1_{t+}\right)dt \leq \overline{p}\left(M+\overline{p}\left(M^0_{t+}-M^1_{t+}\right)dt - M^1_{t+}\right)dt \\ & \leq \overline{p}\left(M-\underline{\gamma}\right)dt. \end{aligned} \tag{38}$$

Therefore, on these states, since $M^0_{t+dt} - M_t$ has to be zero in light of (38), the gain is also zero. To tackle this issue, we decompose the domain of integration in J^{YQ} into three parts based on the (W, M) space. All the following analysis will be on the set of

$$A^1(\varepsilon) = \left\{(W,M) : W < \frac{U(\underline{\gamma})+b}{r} - \varepsilon, M > \underline{\gamma}, \text{and } J_M > 0\right\}$$

where ε is arbitrarily small. Using the results in Section A.2, one can check that on $A^1(\varepsilon)$, $\frac{J_{WW}J_{MM}-J_{WM}}{J_{MM}} = \frac{\widetilde{j}_{ww}\widetilde{j}_{mm}-\left(\widetilde{j}_{wm}\right)^2}{\frac{1}{\gamma^2 r^2}\widetilde{j}_{ww}+\frac{2}{\gamma r}\widetilde{j}_{wm}+\widetilde{j}_{mm}}$ is strictly negative and bounded by $\phi(\varepsilon) < 0$.[27] The other two sets are

$$A^2 = \left\{(W,M) : M = \underline{\gamma} \text{ and } J_M = 0\right\}, A^3(\varepsilon) = \left\{(W,M) : \frac{U(\underline{\gamma})+b}{r} - \varepsilon < W \leq \frac{U(\underline{\gamma})+b}{r}, M > \underline{\gamma}\right\}.$$

We ignore A^2 as the gain is zero. For $A^3(\varepsilon)$, one can show that $M^{*\prime}\left(\frac{U(\underline{\gamma})+b}{r}\right) = 0$, so on $A^3(\varepsilon)$ we only have to consider the region $M \geq \underline{\gamma} - l(\varepsilon)\varepsilon$, where $l(\varepsilon) \to 0$ as $\varepsilon \to 0$. This implies that around this singular point the gain on A^3 is bounded by ε; and by taking $\varepsilon \to 0$, we can focus on the region $A^1(\varepsilon)$.

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[27] To see this, similar to the argument of Cauchy-Schwartz inequality in A.2, one can show that

$$\widetilde{j}_{ww}\widetilde{j}_{mm} - \left(\widetilde{j}_{wm}\right)^2 > -\widetilde{j}^i_{ww}\left[\left(\frac{p}{r+p} - K(m)\right)\left(\frac{b-rw}{b}\right)^{1+\frac{p}{r}} + \frac{r}{r+p}\right]\frac{1}{\gamma r m^2}$$

where $K(m) = \frac{1}{Y\gamma r}\left[\frac{m}{\gamma}-1\right]^2 < 1$. The above term takes a positive bounded-from-below value for $w \leq w^*(m) < \frac{b}{r}$, because each term is bounded away from zero (the claim for $-\widetilde{j}^i_{ww}$ is due to (31)). Also it follows that $J_{MM} = \frac{1}{\gamma^2 r^2}\widetilde{j}_{ww} + \frac{2}{\gamma r}\widetilde{j}_{wm} + \widetilde{j}_{mm} < 0$, because if it is zero, then it must be the case that $\frac{2}{\gamma r}\widetilde{j}_{wm} \geq \frac{2}{\gamma r}\sqrt{\widetilde{j}_{ww}\widetilde{j}_{mm}}$, contradiction.

Denote the measure induced by $\mathbb{E}\left[\int_0^\tau e^{-rt}dt\right]$ on $A^1(\varepsilon)$ to be υ. Because $\int_{A^1(\varepsilon)} d\upsilon \leq \frac{1}{r}$, Holder's inequality yields (notice $\phi(\varepsilon) < 0$)

$$\int_{A^1(\varepsilon)} k_t^2 \sigma^2 \left(\frac{J_{WW}J_{MM} - J_{WM}}{J_{MM}}\right) d\upsilon < \sigma^2 \phi(\varepsilon) \int_{A^1(\varepsilon)} k_t^2 d\upsilon < \sigma^2 \phi(\varepsilon) \frac{\left(\int_{A^1(\varepsilon)} k_t d\upsilon\right)^2}{\int_{A^1(\varepsilon)} d\upsilon} < r\sigma^2 \phi(\varepsilon) \left(\int_{A^1(\varepsilon)} k_t d\upsilon\right)^2,$$

therefore when $\sigma \to \infty$, $\int_{A^1(\varepsilon)} k_t d\upsilon \to 0$. (Otherwise since the gain is bounded, $J^{YQ} < J$ as $\sigma \to \infty$). Therefore, $k_t \to 0$ except on a zero measure set.

A.7.1 The Proof for an Equivalent (but Relaxed) Setup

Based on the above argument, we change our interpretation which relaxes the problem. Essentially, writing dQ into the contract *creates* a convex cost structure for the agent to exert $a = \overline{p}$ relative to $a = p$, and investors can specify $k_t > 0$ without inducing $a = \overline{p}$. When $\sigma \to \infty$, the above result suggests that the convexity (captured by the upper bound of k_t) diminishes. Therefore, we now consider the model with a convex effort cost as a modification of the linear cost specification, where the agent incurs a cost $\left(\frac{b}{p} + k\right)\epsilon$ in taking $a = \overline{p}$, where $k > 0$ is a constant. We then investigate the limiting case where $k \to 0$, i.e., effort cost becomes linear. This relaxes the investors' problem without any cost (instead of using dQ, the convexity comes for free); but we will show that investors' potential gain $\int_{A^1(\varepsilon)} J_M \left(M_{t+dt}^0 - M_t\right) d\upsilon$ goes to zero even in this framework.

Based on this relaxation, we restrict attention to the case where W and W are deterministic (as investors no long need dQ performance). Because the agent's saving motive is determined by *expected* marginal utility, and the investors' value function is concave, this treatment is innocuous. In fact, we can show that, randomization of M in fact *increases* the agent's private savings gain, and in the same time reduces the investors' value—therefore we obtain a stronger result by ruling out randomization. Interested readers can find the proof in Section A.7.2.

Note that the agent's incentive of "shirking and saving" is counteracted by $k > 0$. When $k \to 0$, to prevent "shirking and saving," the private savings gain must converge to zero also. More specifically, fix a starting time 0 and focus on the path without jumps. Suppose the initial marginal utility is M_0 (so consumption level is c_0). Private savings gain going to zero implies that, future consumption without jumps cannot fall below $c_0 - \eta_c$ for some positive η_c (note that (38) implies that the consumption *falling* path must be continuous with a bounded speed $M_{t+dt}^0 - M_t < p\left(\gamma - \underline{\gamma}\right) dt$, which ensures a strictly positive consumption smoothing gain once c_t goes under c_0 strictly). Equivalently, for $\forall t_2 > t_1$, $M_{t_2} < M_{t_1} + \eta$ for some positive η almost surely, where $\eta \to 0$ as $k \to 0$.

Because $J_M \geq 0$, w.l.o.g we restrict $\{M_t\}$ to lie within a band with a width η, i.e., $M_t \in [M_0, M_0 + \eta]$.[28] Denote the gain from time 0 to $t > 0$ as H_t. Let $\widehat{r} = r + p$, and focus on the path without jumps. We have

$$\begin{aligned} H_t &= \mathbb{E}\int_0^t e^{-rs}\left[J(W_s, M_{s+ds}) - J(W_s, M_s)\right]\mathbf{1}_{\{N_s=0\}} = \int_0^t e^{-\widehat{r}s}\left[J(W_s, M_{s+ds}) - J(W_s, M_s)\right] \\ &= \int_0^t e^{-\widehat{r}s}\left\{\left[J(W_{s+ds}, M_{s+ds}) - J(W_s, M_s)\right] - \left[J(W_{s+ds}, M_{s+ds}) - J(W_s, M_{s+ds})\right]\right\} \\ &= e^{-\widehat{r}t}J(W_t, M_t) - J(W_0, M_0) + \widehat{r}\int_0^t J(W_s, M_s)e^{-\widehat{r}s}ds - \int_0^t e^{-\widehat{r}s}\left[J(W_{s+ds}, M_{s+ds}) - J(W_s, M_{s+ds})\right]. \end{aligned}$$

[28] Due to (38), there is no upward jump for $\{M_t\}$; but it might exhibit downward jump. However, $\{M_t\}$ has bounded variation for the existence of the integral. Also, as $J_M > 0$, the suicide strategy of $M_t = M_t' + f(t)$ where $M_t' \in [M_0, M_0 + \eta]$, $f(0) = 0$ and $f'(t) < 0$ is suboptimal. An argument similar to the one presented here can be invoked to rigorously show this claim (see He (2007c)).

For the last term, let $F(s, M_{s+ds}) = J(W_{s+ds}, M_{s+ds}) - J(W_s, M_{s+ds})$, then $F_M(s, M_{s+ds}) = J_{WM} \cdot dW_s$, and

$$\begin{aligned}
& \int_0^t e^{-\widehat{r}s} F(s, M_{s+ds}) = \int_0^t e^{-\widehat{r}s}\left[F(s, M_0) + F_M\left(s, \widetilde{M}\right)(M_{s+ds} - M_0)\right] \\
= & \int_0^t e^{-\widehat{r}s}\left[F(s, M_0) + J_{WM}\left(W_s, \widetilde{M}\right) dW_s (M_{s+ds} - M_0)\right] \\
\geq & \int_0^t e^{-\widehat{r}s}\left[J(W_{s+ds}, M_0) - J(W_s, M_0)\right] - K_1 \cdot \eta \\
= & \ e^{-\widehat{r}t} J(W_t, M_0) - J(W_0, M_0) + \widehat{r}\int_0^t J(W_s, M_0) e^{-rs} ds - K_1 \cdot \eta,
\end{aligned}$$

where the first equality uses the Intermediate Value Theorem with $\widetilde{M} \in (M_0, M_{s+ds})$. The third inequality (where K_1 is a positive constant) requires a bit explanation: without jumps, dW_s is bounded by Bdt; and with bounded J_{WM}, we get an estimate of K_1 to bound the term $J_{WM}\left(W_s, \widetilde{M}\right) dW_s (M_{s+ds} - M_0)$. Therefore,

$$\begin{aligned}
H_t \leq & \ e^{-\widehat{r}t} J(W_t, M_t) - e^{-\widehat{r}t} J(W_t, M_0) + \widehat{r}\int_0^t J(W_s, M_s) e^{-\widehat{r}s} ds - \widehat{r}\int_0^t J(W_s, M_0) e^{-\widehat{r}s} ds + K_1\eta \\
= & \ (K_2 + K_3 + K_1)\eta,
\end{aligned}$$

and these constants K_i's are independent of t, W, and M. Now integrating over all possible jumps, standard Poisson setup implies that the gain is dominated by $K\epsilon$ where K is a constant.

As a summary, when $\sigma \to \infty$, the convexity k that is allowed in the contract goes to zero, which implies that η, the increment of M without a jump allowed in the complete contract, goes to zero. Therefore, as shown above, the contractual gain in (36) converges to zero. This shows that even though it will be beneficial to incorporate dQ information, when its precision drops to zero which is featured by most soft information, the value from the optimal complete contract converges to J derived in the main text.

A.7.2 Ruling out Randomization

We rule out randomization as a value-improving scheme in this section. Note that w.l.o.g, it is never optimal for investors to assign $c_t > \frac{1}{\gamma}\ln\frac{\gamma}{\underline{\gamma}}$, until the first-best states $M = \underline{\gamma}$ and $W \geq \frac{U(\gamma)+b}{r}$ are reached. The reason is simple: to deter saving it is enough to set c_t at the upper bound $\frac{1}{\gamma}\ln\frac{\gamma}{\underline{\gamma}}$, and it is suboptimal to raise c above $\frac{1}{\gamma}\ln\frac{\gamma}{\underline{\gamma}}$ until the first-best state (recall $J_W > -\frac{1}{\underline{\gamma}}$ on $A^1(\varepsilon)$). This implies that, we can restrict attention to the case where $c_t \leq \frac{1}{\gamma}\ln\frac{\gamma}{\underline{\gamma}}$, and as a result $U(M)$ is always linear in M under the form of CARA utility.

Randomization Leads to a Higher (Extra) Savings Gain for the Agent Fixing a time interval $[0, t]$ and the agent decide to shirk and save. Suppose that, w.l.o.g, given a deterministic increasing M_s the optimal consumption (in fact, marginal utility) plan is M_s^*. Therefore

$$dS_t = rS_t dt + \frac{\ln\gamma - \ln M_t}{\gamma} dt + \frac{\ln\gamma - \ln M_t^*}{\gamma} dt = rS_t dt + \frac{\ln M_t^*/M_t}{\gamma} dt.$$

Let the deterministic $M_s^*/M_s = g_s > 0$.

Now consider a random marginal utility process $\widetilde{M_s}$ which has $\mathbb{E}\widetilde{M_s} = M_s$, and consider the following simple rule which mimics the evolution of dS_t up to the nonnegativity constraint of consumption. Specifically, consider the following policy: at t, if there is no $g_s\widetilde{M_s} > \gamma$ for $s < t$ then $\widetilde{M_t^*} = g_t\widetilde{M_t}$; otherwise saves whenever possible to get back the designed saving balance S_t. Clearly this policy performs better than the naive rule $\widetilde{M_t^\Delta} = g_t\widetilde{M_t}$ where c_t could go negative, because the above rule features a better smoothing (to see this, take any path and the extra smoothing gain is from the interval with zero consumption). But clearly $\left\{\widetilde{M}_t^{\Delta}\right\}$ without nonnegative consumption constraint delivers the same utility as M_s, since

$$\mathbb{E}\int_0^t e^{-rs}U\left(\widetilde{M}_s^{\Delta}\right)ds = \int_0^t e^{-rs}U\left(g_s M_s\right)ds = \int_0^t e^{-rs}U\left(M_s^*\right)ds.$$

Because the optimal saving rule results in a higher utility, the agent's saving gain increases under randomization.

Randomization Leads to a Lower Value for Investors Now we show that, due to the concavity of J, randomization in W and M hurts investors, and the reason is just By checking (36), the instantaneous gain (with randomization) is $e^{-rs}J\left(W_{s+ds}, M_{s+ds}\right) - J\left(\mathbb{E}_s W_{s+ds}, M_s\right)$. Focus on the path without jumps; the total gain is

$$X = \mathbb{E}\int_0^t e^{-rs}\left[J\left(W_{s+ds}, M_{s+ds}\right) - J\left(\mathbb{E}_s W_{s+ds}, M_s\right)\right]\cdot\mathbf{1}_{\{N_s=0\}} = \mathbb{E}\int_0^t e^{-\widehat{r}s}\left[J\left(W_{s+ds}, M_{s+ds}\right) - J\left(\mathbb{E}_s W_{s+ds}, M_s\right)\right]$$

Because

$$\int_0^t e^{-\widehat{r}s}\left[J\left(W_{s+ds}, M_{s+ds}\right) - J\left(W_s, M_s\right)\right] = e^{-\widehat{r}t}J\left(W_t, M_t\right) - J\left(W_0, M_0\right) + \widehat{r}\int_0^t e^{-\widehat{r}s}J\left(W_s, M_s\right)ds,$$

we have

$$\begin{aligned}
X &= \mathbb{E}\left[e^{-\widehat{r}t}J\left(W_t, M_t\right) + \int_0^t \left[e^{-\widehat{r}s}\left(-J\left(\mathbb{E}_s W_{s+ds}, M_s\right) + J\left(W_s, M_s\right) + \widehat{r}J\left(W_s, M_s\right)ds\right)\right] - J\left(W_0, M_0\right)\right] \\
&= \mathbb{E}\left[e^{-\widehat{r}t}J\left(W_t, M_t\right) + \int_0^t \left[e^{-\widehat{r}s}\left(-J_W\left(rW_s - U\left(M_s\right) - b\right)dt + \widehat{r}J\left(W_s, M_s\right)ds\right)\right] - J\left(W_0, M_0\right)\right] \\
&= \mathbb{E}\left[e^{-\widehat{r}t}J\left(W_t, M_t\right) + \int_0^t \left[e^{-\widehat{r}s}\left(-c\left(M_s\right) + pY + pJ\left(W_s + \frac{b}{p}, M_s\right)\right)\right] - J\left(W_0, M_0\right)\right]
\end{aligned}$$

where we use the equation in (13) to get the last equality. Now comparing this gain with the one under the deterministic evolutions of state variables $\left(\mathbb{E}W_s, \mathbb{E}M_s\right)$, we have (note that $-c\left(\cdot\right)$ is concave too)

$$\begin{aligned}
X &\leq e^{-\widehat{r}t}J\left(\mathbb{E}W_t, \mathbb{E}M_t\right) + \int_0^t \left[e^{-\widehat{r}s}\left(-c\left(\mathbb{E}M_s\right) + pY + pJ\left(\mathbb{E}W_s + \frac{b}{p}, \mathbb{E}M_s\right)\right)\right] - J\left(W_0, M_0\right) \\
&= e^{-\widehat{r}t}J\left(\mathbb{E}W_t, \mathbb{E}M_t\right) + \int_0^t \left[e^{-\widehat{r}s}\left(-J_W\left(\mathbb{E}W_s, \mathbb{E}M_s\right)\left(r\mathbb{E}W_s - U\left(\mathbb{E}M_s\right) - b\right)dt + \widehat{r}J\left(\mathbb{E}W_s, \mathbb{E}M_s\right)ds\right)\right] - J\left(W_0, M_0\right) \\
&= \int_0^t e^{-\widehat{r}s}\left[J\left(\mathbb{E}W_{s+ds}, \mathbb{E}M_{s+ds}\right) - J\left(\mathbb{E}W_{s+ds}, \mathbb{E}M_s\right)\right]
\end{aligned}$$

where we use the fact that by definition $\mathbb{E}_s W_{s+ds} - W_s = \left(rW_s - U\left(M_s\right) - b\right)dt$ without jump, and for the deterministic states,

$$\left(r\mathbb{E}W_s - U\left(\mathbb{E}M_s\right) + b\right)dt = \mathbb{E}\left[rW_s - U\left(M\right) + b\right]dt = \mathbb{E}\left[dW_s\right] = d\mathbb{E}W_s.$$

This completes the proof.